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RESEARCH ARTICLE

ON THE PROJECTIVE ALGEBRA OF FIRST APPROXIMATE MATSUMOTO METRIC

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ABSTRACT

In the present paper, we have introduce a special metric, called First Approximate Matsumoto metric is analyzed as a certain on an n dimensional space with the projective Algebra and Lie Algebra of the projective group and this metric is characterized as certain Lie sub algebra of the projective algebra. Further, which is devoted to studying the condition of Finsler space of constant flag curvature and vanishing S curvature admits a non Riemannian space of affine projective vector field with First Approximate Matsumoto metric is Berwald space.

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INTRODUCTION

A Finsler metric on a manifold is a family of norms in tangent spaces, which vary smoothly with the base point. Every Finsler metric determines a spray by its systems of geodesic equations. Thus, Finsler spaces can be viewed as special spray spaces. On the other hand, every Finsler metric defines a distance function by the length of minimial curves. Thus Finsler spaces can be viewed as regular metric spaces. Riemannian spaces are special regular metric spaces. In 1854, B. Riemann mentioned general regular metric spaces, but he thought that there were nothing new in the general case. In fact, it is technically much more difficult to deal with general regular metric spaces. For more than half century, there had been no essential progress in this direction until P. Finsler did his pioneering work in 1918. Finsler studied the variational problems of curves and surfaces in general regular metric spaces. Some difficult problems were solved by him. Since then, such regular metric spaces are called Finsler spaces [14].

As the projective algebra of a metric is defined in terms of its geodesic spray and it is defined for any spray and coincides for projectively equivalent sprays. Itis well known that in an n-dimensional Riemannian space of constant curvature the dimensionof $p(M,F)$ is $n(n+2)$ and vice-versa . This weaves an overture for an analogue problem of Randers space. If we have $s^i_j = 0$, then the respective projective algebra of metric F and $p(M,\alpha)$ of α coincide. The important case is considerable when $s^i_j \neq 0$ and uncovers a non-Riemannian feature of Finsler metrics in comparison with the analogue Riemannian case. A projective vector field is said to be C-projective if its projective factor p has closeness property. In the recent works [3] projective algebra of Randers metrics are introduced and the Lie algebras of C- projective vector fields and the well known non- Riemannian curvature and H- curvature is invariantsof the algebras of C- projective vector fields are also introduced. The concept of (α,β) - metric was introduced in 1972 by M.Matsumoto and studied by M.Hachiguchi (1975), Y. Ichijjyo (1975). S. Kikuchi (1979), C.Shibata (1984), Gouree Shankar and Ravindra Yadav (2011), Narasimhamurthy S.K. and Chethan B.C.(2015).[4]The S-curvature is constructed by Shen on Finsler manifolds. A Finsler metric F on an n-dimensional manifold M is said to have isotropic S-curvature if isotopic S= (n +1)c(x)F, for some scalar function c on M . It is known that some of Randers metrics are of S-curvature [6] . The examplesof (α,β) - metrics are Randers metric, Kropina metric, Matsumoto metric. The present paper is devoted to studying the condition for a Finsler space with (α,β) metric $F = \alpha + \beta + \frac{\beta^2}{\alpha}$ is Berwald space.

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PRELIMINARIES

Let M be an n-dimensional smooth manifold and $\pi: TM\backslash\{0\} \to M$ the natural projection from the tangent bundle. Let (x,y) be a point of TM with $x \in M, Y \in TxM$ and let

(x^i, y^i) be the local coordinates on TM with $y = y^i \; \partial/(\partial x^i)$.

Finsler metric on M is a function $F : TM \to [0,\infty)$ satisfying the following properties

- Regularity: $F(x,y)$ is smooth in $TM \backslash\{0\}$
- Positive homogeneity: $F(x,\lambda y) = \lambda F(x,y)$ for $\lambda > 0$.
- Strong concavity: The fundamental quadratic form $g = g_{ij}(x,y)dx^i \otimes dx^j$ is positive definite,

Where

$$g_{ij} = \frac{\partial^2(F)^2}{2\partial y^i \partial y^j}, C_{ijk} = \frac{1}{4}[F]^2{}_{y^i y^j y^k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

Define symmetric trilinear form $C = C_{ijk}dx^i \otimes dx^j \otimes dx^k$ on $TM \backslash\{0\}$. We call C is the Cartan torison. Let F be a Finsler metric on an n - dimensional manifold M. The canonical geodesic $\sigma(t)$ of F is characterized by

$$\frac{d^2\sigma^i(t)}{dt^2} + 2G^i(\sigma(t), \dot{\sigma}(t)) = 0,$$

where G^i are the geodesic coefficients having the expression

$$G^i = \frac{1}{4}g^{ij}\{[F^2]_{x^k y^l}y^k - [F]^2{}_{x^l}\} \text{ with } (g)^{ij} = (g)_{ij}{}^{-1} \text{ and } \dot{\sigma} = \frac{d\sigma^i}{dt}\frac{\partial}{\partial x^i}.$$

A spray on M is a globally C^∞vector field G on $TM\backslash\{0\}$ which is expressed in local coordinates as follows

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}.$$

Assume the conventions:

$$G^i_j = \frac{\partial G^i}{\partial y^i}, G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}, G^i_{jkl} = \frac{\partial G^i_{jk}}{\partial y^l}.$$

Note that G^i_{jk} gives rise to a torison –free connection in π^*TM called the Berwald connection in [5]. The function G^i_j define a non linear connection HTM spanned by horizontal frame $\left\{\frac{\partial}{\partial x^i}\right\}$, where $\frac{\delta}{\delta x^j} - G^i_j \frac{\partial}{\partial y^i}$. The nonlinear connection HTM splits TTM as $TTM = ker\pi_* \oplus HTM$, see[5]. If $G^i_{jk}(x,y)$ are functions of $x \in M$, equivalently at every point of F if and only if $G^i_{jkl} = 0$ then the Finsler metric is called a Berwald metric.

Projective Vector fields on special (α,β) – Metric

In this section, Finsler space of a special (α,β) − metric is analyzed as a certain Lie sub algebra of the projective algebra $p(M,\alpha)$. Further, we proved that this metric is Berwald space.

Let (M,α) be a Riemannian space $\left(\alpha = \sqrt{a_{ij}y^i y^j}\right)$and $\left(\beta = b_i(x)y^i\right)$ on a manifold M then$\|\beta\|_x = sup_{y\in\, _xM} \frac{\beta(y)}{\alpha(y)} < 1$. The Finsler (α,β) −metric, $F = \alpha + \beta + \frac{\beta^2}{\alpha}$is called a First Approximate Matsumoto metric on M. G^i_α and G^i denote the geodesic spray coefficients of α and F and the Levi-Civita connection of α by ∇. Define $\nabla_j b_i$ by $(\nabla_j b_i)\theta^j = db_i - b_j\theta^j_i$, where $\theta^i = dx^i$, $\theta^j_i = \tilde{\gamma}^j_{ik}\, dx^k$ and ∇ is the covariant derivation of α. Let us put

$r_{ij} = 1/2(\nabla_j b_i + \nabla_i b_j)$, $s_{ij} = 1/2\,(\nabla_j b_i - \nabla_i b_j)$,

$s^i_j = a^{ih}s_{hj}$, $\quad s_j = b_i s^i_j$

$e_{ij} = r_{ij} + b_i s_j + b_j s_i$

Clearly, β is closed if and only if $s_{ij} = 0$. Let $s_j = b^i s_{ij}$, $s^i_j = a^{ij}$, $s_0 = s_i\, y^i$, $s^i_0 = s^i_j y^j$, $r_{00} = r_{ij}y^i y^j$. The geodesic coefficients G^i of F and G^i_α of α are related as follows

$$G^i = G^i_\alpha - \frac{1}{b^2}\left(\frac{r_{00}}{F} + s_0\right)y^i - \frac{F}{2}s^i_0$$

Let V is projective vector field on (M,F) then it is Douglas tensor $L_{\hat{V}}D^i_{jkl} = 0$. The sprays G^i of F and $\hat{G}^i = G^i_\alpha + T^i$ and hence

$G^i = G^i_\alpha - \frac{1}{b^2}\left(\frac{r_{00}}{F} + s_0\right) y^i - \frac{F}{2} s^i_0$

$$D^i_{jkl} = \hat{G}^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left\{T^i - \frac{1}{n+1} T^m_{y^m} y^i\right\}$$

$$T^i = \alpha Q s^i_0 + \Psi\{-2Q\alpha s_0 + r_{00}\} b^i \quad (1)$$

$$Q = \frac{\varphi'}{\varphi - s\varphi'} = \frac{1+2s}{1-s^2} \quad (2)$$

$$\Psi = \frac{1}{2}\frac{\varphi''}{(\varphi - s\varphi') + (b^2 - s^2)\varphi''} = \frac{1}{1 - 3s^2 + 2b^2} \quad (3)$$

$$T^i = \frac{1+2s}{1-s^2}\,\alpha\, s^i_0 - \frac{1}{1-3s^2+2b^2}\left\{\left(\frac{1+2s}{1-s^2}\right) 2\alpha s_0 - r_{00}\right\} b^i \quad (4)$$

$$T^m_m = \varphi' s_0 + \Psi' \alpha^{-1}(b^2 - s^2)(r_{00} - 2\varphi\alpha s_0)$$

$$+2\Psi[r_0 - \varphi^1(b^2 - s^2)s_0 - \varphi s s_0] \quad (5)$$

On calculation, $T^m_m = 0$.

From this and take $L_{\hat{v}}\left\{\frac{\alpha(1+2s)}{1-s^2}\, s^i_0\right\} = 0$, then we have

$$L_{\hat{v}}\, D^i_{jkl} = L_{\hat{v}}\, T^i_{.j.k.l} = L_{\hat{v}}\, \left\{\frac{\alpha(1+2s)}{1-s^2}\, s^i_0\right\}_{.j.k.l} = 0$$

Therefore, we have $H^i(x,y), (i = 1,2,3, \dots n)$ is quadratic in y then

$$L_{\hat{v}}\, \left\{\frac{\alpha(1+2s)}{1-s^2}\, s^i_0\right\} = H^i \quad (6)$$

Now, let us take $t_{ij} = L_{\hat{v}}\, a_{ij}$.

Observe that $L_{\hat{v}}\, \left\{\frac{\alpha(1+2s)}{1-s^2}\, s^i_0\right\} = \frac{t_{00}}{2\alpha}\, s^i_0 + \frac{\alpha(1+2s)}{1-s^2}\, L_{\hat{v}}\, s^i_0$ (7)

Using (7), (6) can be written as

$$t_{00}\, s^i_0 + 2\alpha^2(1+2\alpha) L_{\hat{v}} s^i_0$$
$$= (1 - s^2) H^i . \quad (8)$$

Here we see that $\alpha^2 = a_{ij}(x) y^i y^j$, $t_{00}\, s^i_0 = \left(t_{ij}(x) s^i_k(x) y^i y^j y^k\right)$ and

$L_{\hat{v}}\, s^i_0 = \left(L_v s^i_k\right)(x) y^k$ are polynomials in $y^1, y^2 \dots . y^n$.

Hence, the l h s of (8) is a polynomial in $y^1, y^2 \dots . y^n$ for all i,

but the r h s is not . It follows that $H^i = 0$ for all i , (7) leads

as $L_{\hat{v}}\left\{\frac{\alpha(1+2s)}{1-s^2}\, s^i_0\right\} = 0$.

From [2] geodesic coefficients of F are investigated as follows

$$G^i_\alpha = G^{a\,i}_{\ \alpha} - \frac{\alpha(1+2s)}{1-s^2} - \left(\frac{\alpha(1+2s)}{1-s^2} s_0 - r_{00}\right)\frac{1}{1-3s^2+2b^2}\left\{b^i + \alpha^{-1}\left(\frac{2+5s+5s^2}{2(1+s+s^2)(1-3s^2+2b^2)}\right) y^i\right\}$$

$$G^i_\alpha = G^{a\,i}_{\ \alpha} - \left(\frac{\alpha(1+2s)}{1-s^2} s_o - r_{00}\right)\left(\frac{2+5s+5s^2}{2(1+s+s^2)(1 = 3s + 2b^2)}\right)\alpha^{-1} y^i + \frac{\alpha(1+2s)}{1-s^2} s^i_o \ . \quad (9)$$

Since $L_{\hat{v}}\, \left\{\frac{\alpha(1+2s)}{1-s^2} s^i_0\right\} = 0$ and $L_{\hat{v}}\, G^i = p y^i$, from this we have

$$L_{\hat{v}}\, G^i_\alpha = L_{\hat{v}}\, \left\{G^i_\alpha - \left(\frac{\alpha(1+2s)}{1-s^2} s_0 - r_{00}\right)\left(\frac{2+5s+5s^2}{2(1+s+s^2)(1-3s+2b^2)}\right)\alpha^{-1} y^i\right\} = p y^i$$

and finally we obtain

$$L_{\hat{v}}\, G^i_\alpha = L_{\hat{v}}\, \left\{p + \left(\frac{\alpha(1+2s)}{1-s^2} s_0 - r_{00}\right)\left(\frac{2+5s+5s^2}{2(1+s+s^2)(1-3s+2b^2)}\right)\alpha^{-1}\right\} y^i .$$

It shows that, the vector field V is (M, F) .

Conversely, suppose V is (M, F) i.e., $L_{\hat{v}} = w_0 y^i$ for some $1-$form, $w_0 = w_k(x) y^k$ on M and $L_{\hat{v}} \left\{\frac{\alpha(1+2s)}{1-s^2} s_0^i\right\} = 0$.

From (3.9) it follows

$$L_{\hat{v}}\, G^i = L_{\hat{v}} \left\{G_\alpha^i - \left(\frac{\alpha(1+2s)}{1-s^2} s_0 - r_{00}\right)\left(\frac{2+5s+5s^2}{2(1+s+s^2)(1-3s+2b^2)}\right)\alpha^{-1} y^i + \frac{\alpha(1+2s)}{1-s^2} s_0^i\right\},$$

$$L_{\hat{v}}\, G^i = L_{\hat{v}} G_\alpha^i - L_{\hat{v}} \left\{G_\alpha^i - \left(\frac{\alpha(1+2s)}{1-s^2} s_0 - r_{00}\right)\left(\frac{2+5s+5s^2}{2(1+s+s^2)(1-3s+2b^2)}\alpha^{-1}\right)\right\} y^i .$$

$$L_{\hat{v}}\, G^i = \left\{w_0 - L_{\hat{v}} \left\{G_\alpha^i - \left(\frac{\alpha(1+2s)}{1-s^2} s_0 - r_{00}\right)\left(\frac{2+5s+5s^2}{2(1+s+s^2)(1-3s+2b^2)}\alpha^{-1}\right) y^i\right\}\right\}.$$

Which implies that V is (M, F) . Hence we state the following

Theorem 1. Let $\left(M, F = \alpha + \beta + \frac{\beta^2}{\alpha}\right)$ be a special (α, β)-metric and V be a vector field on M. Then V is $F-$ projective if and only if V is (M, α) and $L_{\hat{v}} \left\{\frac{\alpha(1+2s)}{1-s^2}\, s_0^i\right\} = 0$.

By theorem (1) and observe that $L_{\hat{v}} \left\{\frac{\alpha(1+2s)}{1-s^2}\, s_0^i\right\} = 0$.

Let us suppose $t_{ij} = L_{\hat{v}}\, a_{ij}$ and $L_{\hat{v}} \left\{\frac{\alpha(1+2s)}{1-s^2}\, s_0^i\right\} = 0$ and $s_j^i \neq 0$.

Now, let us take $t_{ij} = L_{\hat{v}}\, a_{ij}$ and $L_{\hat{v}} \left\{\frac{\alpha}{1-2s}\, s_0^i\right\} = 0$.

Therefore (8) becomes $\{t_{00}\, s_0^i + 2\alpha^2 L_{\hat{v}} s_0^i\} = 0$ (10)

It follows that α^2 divides $t_{00}\, s_0^i$ for all i. This is equivalent to $s_j^i = 0$. Which contradicts since $s_j^i \neq 0$.

Therefore V is conformal vector field on (M, α).

That is $\alpha-$ projective then there is a constant μ such that $L_{\hat{v}} a_{ij} = 2\mu a_{ij}$.

From (10) we obtain $L_v s_j^i = -\mu s_j^i$.

Observe that
$$\begin{aligned} L_v s_{ij} &= (L_v a_{ik}) s_j^k, \\ &= (L_v a_{ik}) s_j^k + a_{ik}\, L_v\, s_j^k, \\ &= 2\mu s_{ij} - \mu s_{ij}, \\ &= \mu s_{ij}, \end{aligned}$$

It shows that $L_{\hat{v}} d\beta = \mu d\beta$. Hence we state the following

Lemma 1. Let $\left(M, F = \alpha + \beta + \frac{\beta^2}{\alpha}\right)$ be a special (α, β)- metric on an $n-$dimensional. If $s_j^i \neq 0$ then the vector field V is (M, F) if and only if V is a (M, α) homothety and $L_{\hat{v}} d\beta = \mu d\beta$, where $L_{\hat{v}} a_{ij} = t_{ij} = 2\mu a_{ij}$.

Remark 1. From lemma (1), since $s_j^i = 0$, then V is (M, F) projective vector filed, but it is not a $\alpha-$homothety.

Since F is a vanishing $S-$ curvature [1] and (M, α).

From these we have $\alpha^\Psi r_{00} = 2\sigma(x)[\beta^2 - (1-\alpha^\Psi) - \alpha(2\beta - 1)]$.

If the function Ψ and $\Psi(x, y)$ is linear w. r .t. y then we have $L_{\hat{v}}\, G^i = \Psi\, y^i$.

Applying theorem (1) we have,

$$L_{\hat{v}} G^i = V_{\hat{v}} \tilde{G}^i + L_{\hat{v}} \left(\sigma\left(\frac{\beta^2(1-\alpha^\Psi) - \alpha(2\beta-1)}{\alpha^\Psi y^i}\right)\right) - L_{\hat{v}} s_0 y^i = \Psi y^i.$$

Putting $t_{ij} = L_{\hat{v}} a_{ij}$ and $L_{\hat{v}} = 0$, it implies that $t_{00} = L_{\hat{v}} \alpha^2$ and

$$L_{\hat{v}} \tilde{G}^i + \left\{\frac{\beta^2(1-\alpha^\Psi) - \alpha(2\beta-1)}{\alpha^\Psi}\right\} L_{\hat{v}} \sigma y^i + \frac{t_{00}}{2\alpha} c y^i - L_{\hat{v}} s_0 y^i = \Psi y^i. \quad \text{.................................... (11)}$$

Recall that the natural coordinates system $(x^i, y^i), \varphi^{-1}(U)$ and $x \in U$, we treating the above equation as a polynomial in $y^1, y^2, \ldots\, y^n$. Multiplying (11) by α. Then we obtain,

$$Rat^i + \alpha\, Irrat^i = 0\,,\ i = 1,2, \dots n,$$

where

$$Rat^i = \alpha^2 L_{\hat{V}}\, \sigma y^i + \frac{1}{2}\mu a_{ij}\sigma y^i$$

$$Irrat^i = L_{\hat{V}}\, \tilde{G}^i - \left(\left[\frac{\beta^2(1-\alpha^{\Psi}) - \alpha(2\beta-1)}{\alpha^{\Psi}}\right] L_{\hat{V}}\sigma + L_{\hat{V}} s_0 - \Psi\right) y^i.$$

Now, we assume that $s = 0$. By Lemma (1), (M, F) must be locally projectively flat, otherwise V is $\alpha -$ homothety. Which is a contradiction to that V is $\alpha -$ homothety.

Hence $s_{ij} = 0$ and by $e_{ij} = r_{ij} + b_i s_j + b_j s_i$, which implies $r_{ij} = 0$. Which is equivalent to $\nabla_i b_j = 0$ and (M, F) is a Berwald space. Hence we have

Theorem 2. Let $\left(M, F = \alpha + \beta + \frac{\beta^2}{\alpha}\right)$ be a special (α, β)-metric of vanishing $S -$ curvature. If (M, F) admits a (M, α) then it is a Berwald space.

CONCLUSION

In this paper, we studied the projective algebra of (α, β) - metric $F = \alpha + \beta + \frac{\beta^2}{\alpha}$ of constant flag curvature and vanishing S- curvature admits a non α - affine projective vector field is Berwald space. Also, we discussed some results on First Matsumoto space.

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