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RESEARCH ARTICLE

AFFINELY INDEPENDENT SOLUTIONS BASED ALGORITHM FOR THE DICYCLE POLYTOPE

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ABSTRACT

In this paper, we consider the polytope *P(G)* of all elementary dicycles of the digraph *G*. Using the concept of affinely independence, we show how to construct elementary dicycles that incidence vectors are affinelyindependent. This technique is therefore applied to determine the already known dimension of the polytope *P(G)*.

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INTRODUCTION

Let $G = (V,A)$ be a connected digraph with V as vertex set and A as arc set. We mean by dicycle a sequence $(v_0, a_1, \ldots, a_k, v_k)$where k is an integer, $v_0, v_1, \ldots v_k$ are vertices such that $v_0 = v_k$. For every index i, a_i is an arc connecting vertices v_{i-1} and v_i(where $i \in \{1, \ldots k\}$) and, finally, all arcs a_i have the same direction. An elementarydicycleis a directed cycle $(v_0, a_1, \ldots, a_k, v_k)$ in which each vertex v_i, for every index belonging to $\{0 \ldots k\}$, appears once. We denote by $P(G)$ the polytope of all elementary dicycles of G. That is, the convex hull of the set of incidence vectors of elementary dicycles of the digraph G. Thus, $P(G) = conv \{x \in \{0, 1\}A$: *x is an incidence vector of an elementary dicycle*$\}$. The polytope $P(G)$ has been already studied by Balas & Oosten, (2000). The authors present a linear description of the cycle polytope in digraphs. They study the facial structure of valid inequalities defining the polytope $P(G)$. Balas & Stephan, (2009) consider the dominant of the polytope $P(G)$ and derive other facet-defining inequalities for$P(G)$. Hartmann & Ozlukb, (1979) provide a polyhedral analysis of the p-cycle polytope, which is the convex hull of incidence vectors of all the p-elementary dicycles with p arcs of the complete directed graph G. In the case of undirected graphs, the cycle polytope has been studied by Coullard & Pulleyblank, (1989), and after Bauer, (1997). Kovalev *et al*., (1997) and Bauer *et al*., (1998) study the cardinality constrained cycle polytope which is the convex hull of all cycles with at most p nodes on a complete undirected graph. The p- cyclepolytope has been also studied by Nguyen &Maurras, (2001)Nguyen &Maurras, (2002) for $p = 3$ and for $2 < p < n$. Note that cycles in graphs or digraphs play an important role in many applications. One of the most in-teresting and important applications has to do with testing circuits. A circuit can be modeled by a directed graph where the vertices represent gates (which compute boolean functions) and the arcs which represent the wires which connect gates (see, Leiserson & Saxe, (1991). In literature one can find other applications of cycle problem in other areas as analysis of electrical networks, analysis of chemical and biological pathways. For some examples of cycle problem applications, we refer the reader to Serafini & Ukovitch, (1989), Bollob`as, (2002) and Kavitha *et al*., (2009). In this paper, we address a constructive algorithm that generates elementary dicycles of P (G) that incidence vectors are affinely independent. Based on this algorithm, we determine the already known dimension of the polytope P(G), (see, Balas & Oosten, (2000), Balas & Stephan, (2009)). In the rest of this section, we give further definitions and notations. Consider a loopless complete digraph $G = (V,A)$, with $V = \{v_1, v_2, \ldots, v_n\}$and $A = \{a_1, a_2, \ldots, a_m\}$.$n$ and m are vertex and arc numbers of G, respectively. As G is complete, we have $m = n(n - 1)$.

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Given a sub-digraph, say, H, we denote by $A(H)$, its arc set. Particularly, an elementary directed cycle C has $A(C)$ as arc set. We mean by a minimal elementary dicycle$C$ an elementary dicycle which has only two arcs. We recall from the definition of affine independence that vectors γ_i, $i = 1, \ldots, q$ are said affinely independent, if there exists some coefficients λ_i, $i = 1, \ldots, q$ such that the unique solution of systems $\sum_{i=0}^{q} \lambda_i \gamma_i = 0$ and $\sum_{i=0}^{q} \lambda_i = 0$ is$\lambda_i = 0$, $i = 1, \ldots, q$. In the sequel, we denote by $P^{v}_{s,t}$ a s − t elementary dipath with $A(P^{v}_{s,t}) = \{(s, v), (v, t)\}$ 2 Construction of dicycles with affinely independent vectors We introduce an algorithm that constructs elementary dicycles with affinely independent incidence vectors. After, we apply the algorithm to determine the dimension of the dicyclepolytope $P(G)$.

Affinely independent dicycle vectors algorithm: In this paragraph, we describe a constructive algorithm for the elementary dicycle problem. Before, let see the following result.

Lemma 1.Consider a set$\{Ci, i = 1, \ldots, q\}$of some elementary dicycles that incidence vectors γ_i, with $i \in \{1, \ldots, q\}$ are affinely independent. Let $C_{q+1} \notin \{Ci, i = 1, \ldots, q\}$, with incidence vector$C_{q+1}$, be an elementary dicycle that contains an arc $a_j \in A(C_{q+1})$ such that $a_j \notin A(Ci)$, $\forall Ci \in \{Ci, i = 1, \ldots, q\}$. Then, vectors $\gamma_1, \gamma_2, \ldots, \gamma_q, \gamma_{q+1}$are affinely independent.

Proof. The proof follows directly from the definition of affine independence. Given the loopless complete digraph $G = (V,A)$, consider an hamiltonian elementary dicycle, say C_1, with

$$A(C_1) = \{a_1 = (v_1, v_2), a_2 = (v_2, v_3), \ldots, a_{n-1} = (v_{n-1}, v_n), a_n = (v_n, v_1)\}$$

and the minimal dicycleC_2 that pass by the arc $a_1 = (v_1, v_2)$. That is

$$A(C_2) = \{a_1 = (v_1, v_2), a_{n+1} = (v_2, v_1)\}.$$

Note that both dicycles C_1 and C_2pass by the arc $a_1 = (v_1, v_2)$.Let partition the arc set A as follows

$$A = A' \cup A_1 \cup A_2,$$

where$A' = A(C_1) \cup A(C_2)$, $A_1 = \{a_{n+2}, \ldots, a_{n^2-3n+4}\}$and $A_2 = \{a_{n^2-3n+5}, \ldots, a_{n^2-n}\}$.

$A(C_1)$ and $A(C_2)$have been defined above. A_2 is the set of arcs that do not belong to any elementary dicycle that passes by the arc $a_1 = (v_1, v_2)$. Without taking into account the arc (v_1, v_2), as $(n − 2)$ arcs outgoing from vertex v_1 and$(n − 2)$ arcs incoming to vertex v_2, we have$|A_2| = 2n − 4$. This implies that$|A_1| = n2 − 4n + 3$. Indeed

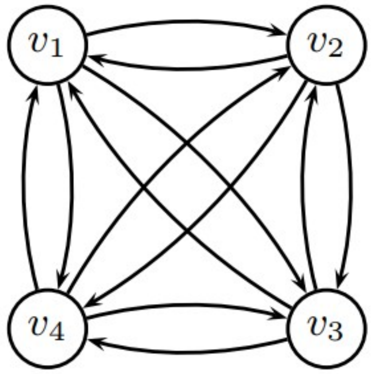
$$n(n − 1) − (n + 1) − (2n − 4) = n2 − 4n + 3.$$

In what follows, let P_{v_1,v_k} , $k \in \{3, \ldots, n\}$be a v_1–v_k elementary dipath such that

$$A(P_{v_1,v_k}) \subset A(C_1).$$

Based on Lemma 1 and with respect to partitions of the arc set A, the following algorithm constructs elementary dicycles that incidence vectors are affinely independent.

Proof. Steps 5 − 8, 11 − 22 and 25 − 30 of Algorithm 1 create respectively $(n − 1)$, $(n^2− 5n + 6)$ and $(2n − 4)$ elementarydicycles. In addition to dicyclesC_1 and C_2, we verify that Algorithm 1 creates $(n − 1)^2$elementary dicycles. On the other hand, at each step, the current created dicycle C_l contains an arc a_i, $i \in \{n + 2, \ldots, n^2− 3n + 4\}$for $a_i \in A_1$ or an are a_i, $i \in \{n^2− 3n + 5, \ldots, n(n − 1)\}$for $a_i \in A_2$ that do not belong to any of the previously createddicycles$C_1, C_2, \ldots, C_{l-1}$. Therefore, according to Lemma 1, incidence vectors of $(n−1)^2$dicycles created by Algorithm 1 are affinely independent.



Algorithm 1. Computation of dieyeles with affinely independent veetors

Data: $G = (V, A)$ A loopless complete directed graph, with $|V| = n$.
Result: Set $\mathcal{C}$ of dicycles with affinely independent incidence vectors

1 **begin**
2 $\mathcal{C} \leftarrow \{C_1, C_2\}$, $a_1 \leftarrow (v_1, v_2)$
3 $l \leftarrow 3$
4 $i \leftarrow n + 2$
5 **for** $k \leftarrow 3$ *To* $(n-1)$ **do**
6 $C_l \leftarrow P_{v_1,v_k} \cup \{(v_k, v_1)\}$ // $a_i = (v_k, v_1) \in A_1$
7 $\mathcal{C} \leftarrow \mathcal{C} \cup \{C_l\}$; $l \leftarrow l + 1$, $i \leftarrow i + 1$
8 **end**
9 $l \leftarrow n$
10 $i \leftarrow 2n - 1$
11 **for** $j \leftarrow 2$ *To* n **do**
12 **for** $k \leftarrow 3$ *To* n **do**
13 **if** $k \neq j + 1$ *and* $j < k$ **then**
14 $C_l \leftarrow P_{v_1,v_j} \cup \{(v_j, v_k)\} \cup \{(v_k, v_1)\}$ // $a_i = (v_j, v_k) \in A_1$
15 $\mathcal{C} \leftarrow \mathcal{C} \cup \{C_l\}$, $l \leftarrow l + 1$, $i \leftarrow i + 1$
16 **end**
17 **if** $k \neq j + 1$ *and* $j > k$ **then**
18 $C_l \leftarrow P^{v_2}_{v_1,v_j} \cup \{(v_j, v_k)\} \cup \{(v_k, v_1)\}$ // $a_i = (v_j, v_k) \in A_1$
19 $\mathcal{C} \leftarrow \mathcal{C} \cup \{C_l\}$, $l \leftarrow l + 1$, $i \leftarrow i + 1$
20 **end**
21 **end**
22 **end**
23 $l \leftarrow n^2 - 4n + 6$
24 $i \leftarrow n^2 - 3n + 5$
25 **for** $k \leftarrow 3$ *To* n **do**
26 $C_l \leftarrow \{(v_1, v_k)\} \cup \{(v_k, v_1)\}$ // $a_i = (v_1, v_k) \in A_2$
27 $l \leftarrow l + 1$, $i \leftarrow i + 1$
28 $C_l \leftarrow \{(v_k, v_2)\} \cup \{(v_2, v_k)\}$ // $a_i = (v_k, v_2) \in A_2$
29 $\mathcal{C} \leftarrow \mathcal{C} \cup \{C_l\}$, $l \leftarrow l + 1$, $i \leftarrow i + 1$
30 **end**
31 **return** $\mathcal{C}$
32 **end**

In this example, we apply Algorithm 1 based on Lemma 1 to construct elementary directed cycles that corresponding incidence vectors are affinely independent. We first compute the hamiltoniandicycleC_1 with $A(C_1) = \{a_1 = (v_1, v_2), a_2 = (v_2, v_3), a_3 = (v_3, v_4), a_4 = (v_4, v_1)\}$ and the minimal dicycleC_2 with $A(C_2) = \{a_1 = (v_1, v_2), a_5 = (v_2, v_1)\}$. Consider partitions A', A_1 and A_2 of A, with $A' = A(C_1) \cup A(C_2)$. As $A_2 = \{a_9 = (v_1, v_3), a_{10} = (v_3, v_2), a_{11} = (v_1, v_4), a_{12} = (v_4, v_2)\}$is the set of arcs by which do not pass any elementary directed cycle that contains the arc$a_1 = (v_1, v_2)$, we set $A_1 = \{a_6 = (v_3, v_1), a_7 = (v_2, v_4), a_8 = (v_4, v_3)\}$. Indeed $A_1 = A \setminus (A' \cup A_2)$. We verify that dicyclesC_1 and C_2 contains $(n + 1) = 5$ distinct arcs,$|A_1| = n^2 - 4n + 3 = 16 - 16 + 3 = 3$ and $|A_2| = 2n - 4 = 2 * 4 - 4 = 4$. Let $C = \{C_1, C_2\}$.

• First iteration of Steps 5-7. Set $l \leftarrow 3$, $i \leftarrow 6$ and $k \leftarrow 3$.

From the arc $a_6 = (v_3, v_1) \in A_1$, we create $C_3 \leftarrow P_{v_1,v_k} \cup \{(v_k, v_1)\}$. That is $A(C_3) = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$. One can verify that $a_6 = (v_3, v_1) \notin A(C_1)$ and $a_6 = (v_3, v_1) \notin A(C_2)$. According to Lemma 1, incidence vectorsγ_1 of C_1,γ_2 of C_2 andγ_3 of C_3 are affinely independent. Reset $C = C \cup \{C_3\}$ and $k \leftarrow 4$.

• First iteration of Steps 11-22. We have $l \leftarrow 4$, $i \leftarrow 7$, $j = 2$, $k = 4$ From the arc $a_7 = (v_2, v_4) \in A_1$, we create $C_4 \leftarrow P_{v_1,v_j} \cup \{(v_j, v_k)\} \cup \{(v_k, v_1)\}$. So, $A(C_4) = \{(v_1, v_2), (v_2, V_4), (v_4, v_1)\}$. Verify that $a_7 = (v_2, v_4) \notin A(C_1)$,$a_7 = (v_2, v_4) \notin A(C_2)$ and $a_7 = (v_2, v_4) \notin A(C_3)$. This implies that inci- dence vectors$\gamma_1$of C_1, $\gamma_2$of C_2, γ_3 of C_3 and γ_4 of C_4 are affinely independent. We Reset $C = C \cup \{C_4\}$, $k \leftarrow 5$.

• **Second iteration of Steps** 11-22.$i \leftarrow 8, j = 4, k = 3$. W.r.t. arc $a_8 = (v_4, v_3) \in A_1$, we create $C_5 \leftarrow P^{v_2}_{v_1,v_j} \cup \{(v_j , v_k)\} \cup \{(v_k, v_1)\}$. So, $A(C5) = \{(v_1, v_2), (v_2, v_4), (v_4, v_3), (v_3, v_1)\}$. C_5contains arcs $a_1 = (v_1, v_2)$ and $a_8 = (v_4, v_3)$. We find $a_8 = (v_4, v_3) \notin A(Cj), j = 1, . . . , 4$. Therefore, according to Lemma 1, incidence vectors γ_l of C_l, with $l = 1, . . . , 5$ are affinely independent. We Reset $C \leftarrow \{C_1,C_2,C_3,C_4,C_5\}$.

•**First iteration of Steps** 25-30. Set $l \leftarrow 6, i \leftarrow 9$ and $k \leftarrow 3$. From $a_9 = (v_1, v_3) \in A_2$ and $a_{10} = (v_3, v_2)$, we create C_6 and C_7 with $A(C_6) = \{(v_1, v_3)(v_3, v_1)\}$and $A(C_7) = \{(v_3, v_2)(v_2, v_3)\}$, respectively. We have $a_9 = (v_1, v_3) \notin A(C_l), l = 1, . . . , 5$ and $a_{10} = (v_3, v_2) \notin A(C_l), l = 1, . . . , 6$. Therefore its corresponding incidence vectors $\gamma_l, l = 1, . . . , 7$ are affinely independent. Reset $C \leftarrow \{C_l, l = 1, . . . , 7\}$and $l \leftarrow 8$.

• **Second iteration of Steps** 25-30. $i \leftarrow 11, k \leftarrow 4$. From $a_{11} = (v_1, v_4) \in A_2$ and $a_{12} = (v_4, v_2)$, we create C_8 and C_9 with $A(C_8) = \{(v_1, v_4)(v_4, v_1)\}$and $A(C_9) = \{(v_4, v_2)(v_2, v_4)\}$, respectively. We have $a_{11} = (v_1, v_4) \notin A(C_l), l = 1, . . . , 7$ and $a_{12} = (v_4, v_2) \notin A(C_l), l = 1, . . . , 8$. Therefore its corresponding incidence vectors $\gamma_l, l = 1, . . . , 9$ are affinely independent. Reset$C \leftarrow \{C_l, l = 1, . . . , 9\}$.

• **The algorithm terminates and returns** $C = \{C_l, l = 1, . . . , 9\}$with $i \leftarrow 13 > m$ and $k = 5 > n$. W.r.t the loopless and complete digraph of Figure 1, by applying Algorithm 1, we create $(n - 1)^2 = 32 = 9$ elementarydicycles that incidence vectors are affinely independent. Below, we apply Algorithm 1 to determine the dimension of dicylepolytope $P(G)$.

Dimension of the dicyclepolytope *P(G):* Balas and Oosten (2000) have shown that the dimension of the dicyclepolytope $P(G)$ is $(n-1)^2$. They first prove thatthe polytope $P(G)$ is the projection of a special case of the prize collecting traveling salesman polyhedron. Thus, from the dimension of the latter polyhedron, they deduce that the one of the cycle polytope $P(G)$ is $(n - 1)^2$. Here, in a different approach, to determine the dimension of $P(G)$, we mainly apply Algorithm 1 defined above and resort to results of following lemmas.

Lemma 2. If $n > 4$, there exists a pair of arcs $a_i = (v_j ,v_k) \in A_1$ and $a_{i'} = (v_k, v_j) \in A_1$ such that by applying Algorithm 1, only unique and distinct dicycles, say C_l and$C_{l'}$, contain arcs $a_i = (v_j , v_k) \in A_1$ and $a_{i'} = (v_k, v_j) \in A_1$, respectiveley.

Proof. If $n \leq 4$, one can easily verify that such a pair of arcs do not exist. So, from Steps 14 and 18 of Algorithm 1, it is clear that arcs of type $(v_k, v_1) \in A_1, k = 3, . . . , n - 1$ and arcs of type $(v_2, v_j) \in A_1, j = 4, . . . , n$ have been used to create several elementary dicycles. However, arcs $a_i = (v_j ,v_k) \in A_1$ and$a_{i'} = (v_k, v_j) \in A_1$, with $v_j \neq v_1, v_2$ and $v_k \neq v_1, v_2$, are used once to create dicycles, sayC_l and $C_{l'}$. That is, only dicycles C_l and $C_{l'}$ contain arcs $a_i = (v_j , v_k) \in A_1$ and $a_{i'} = (v_k, v_j) \in A_1$, respectively, with $v_j \neq v_1, v_2$ and $v_k \neq v_1, v_2$. Indeed, values of k and j change in Steps 14 and 18.

Lemma 3. Consider the set C of elementary dicycles $C_l, l = 1, . . . , (n - 1)^2$ obtained by applying Algorithm 1. Let $C_{l''}$ be the minimal dicycle formed by arcs $a_i = (v_j ,v_k) \in A_1$ and $a_{i'} = (v_k, v_j) \in A_1$ defined in Lemma 2. Incidence vectors of dicycles of the set $C \cup \{C_{l''}\}$ are affinely independent.

Proof. We know that incidence vectors $\gamma_i, i = 1, . . . , (n - 1)^2$of elementary dicycles $C_i, i = 1, . . . , (n - 1)^2$ of C, created by applying Algorithm 1, are affinely independent. Let show that incidence vectors of dicycles of the set $C \cup \{C_{l''}\}$are also affinely independent. Consider the minimal elementary dicycle $C_{l''}$, (with $\gamma_{l''}$as incidence vector and $A(C_{l''}) = \{a_i, a_{i'}\}$). According to Lemma 2, there exists arcs a_i and $a_{i'}$ such that among all other dicycles of $C \cup \{C_{l''}\}$, only dicycles $C_l, C_{l''}$ commonly contain the arc a_i and only dicycles$C_{l'}, C_{l''}$ commonly contain the arc $a_{i'}$. So, applying the affine independence definition to the set $C \cup \{C_{l''}\}$, w.r.t. arcs a_i and $a_{i'}$, we can write the following equations $\lambda_l + \lambda_{l''} = 0$ and$\lambda_{l'} + \lambda_{l''} = 0$, respectively. On the other hand, in opposite To dicycles C_l and $C_{l'}$, the minimal cycle $C_{l''}$do not contain the common arc $a_1 = (v_1, v_2)$ that belongs to all dicycles of the arc set C created from Algorithm 1. This implies that $\lambda_l = \lambda_{l'} = \lambda_{l''} = 0$. It then follows that all $\lambda_i = 0, i = 1, . . . , (n - 1)^2$ and $\lambda_{l''} = 0$ showing that the incidence vectors of elements of $C \cup \{C_{l''}\}$ are affinely independent. If $n = 4$, as there is no arcs $a_{i'} \in A_1$ and $a_{i''} \in A_1$ that can form the minimal dicycle $C_{l''}$, one can consider $C_{l''}$ with $A(C_{l''}) = \{(v_3, v_4), (v_4, v_3)\}$. Refering to Example 1 described above, only the hamiltoniandicycleC_1 contains the arc (v_3, v_4) and only the dicycleC_5contains (v_4, v_3). However, note that $(v_3, v_4) \notin / \in A_1$.

Theorem 2. The dimension of $P(G)$ is $(n - 1)^2$ with $n \geq 3$.

Proof. By virtue of Theorem 1 and Lemma 3, it's possible to create $((n - 1)^2 + 1)$ elementary dicycles $C_l, l = 1, . . . , (n-1)^2+1$, with $\lambda_l, l = 1, . . . , (n-1)^2+1$ as incidence vectors such that vectors $\lambda_1, \lambda_2, . . . , \lambda_{(n-1)^2}, \lambda_{(n-1)^2+1}$are affinely independent. This completes the proof.

CONCLUSION

In general, in combinatorial optimization and particularly in polyhedral theory to determine the dimension of a polyhedron, one has to look for the rank of the affine subspace of the polyhedron. The main contribution of this paper is to address an algorithm that generates elementary dicycles with affinely independent incidence vectors. After, to show its usefulness, applying the algorithm, we determine the dimensionof elementary dicyclepolytope unlike the traditional approach that consists to determine the rank of the affinesubspace of the polyhedron. Note also that such an algorithm can be adapted to discuss the facetness of a given valid inequality of the elementary dicyclepolytope.

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