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RESEARCHARTICLE

REPRESENTATION OF $C\phi_{k,h}$ (Q)

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ABSTRACT

In this paper, we obtain the Hardy – Ramanujan – Rademacher series for $c\phi_{k,h}(n)$ on the lines of L.W. Kolitsch. The existence of such series for $c\phi_{1,k}(n)$ and $c\phi_{k,1}(n)$ was asked for by Andrews and later obtained by Kolitsch. Finally we extend the results on q-binomial coefficients and q-series representation of Andrews to our function $c\phi_{k,h}(n)$. Andrews has established the two congruences $c\phi_{1,2}(5n+3) \equiv c\phi_{2,1}(5n+3) \equiv O \pmod 5$. We show that the analogous congruence $c\phi_{2,2}(5n+3) \equiv O \pmod 5$ is false for n = 2. We also study generalised Frobenius partitions with some restriction on its parts.

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INTRODUCTION

In this paper we consider the problem of representing $C\phi_{k,h}(q)$ as sum of infinite products for arbitrary positive integers k and h. we show that this is possible by generalising the methods of this paper which we do through Lemmas 1-4. Lemma 1 is used in proving Lemma 2 which is essential in obtaining the Laurent expansion of a more general product. The Laurent expansion is obtained in Lemma 3. Lemma 4 furnishes a result which plays the role played by (i) Jacobi's triple product identity. Due to the mechanical nature of the steps we only sketch our proofs and avoid lengthy expressions.

Some Lemmas:

Lemma 1. For a, b, c arbitrary integers, z, α, β non-zero, |q| < 1 and S any function of β, z, q, m

$$\sum_{m=-\infty}^{\infty} S(\beta, z, q, m) \sum_{n=-\infty}^{\infty} q^{an^2-bmn+cn}\alpha^n$$
$$= (q^{2a}; q^{2a})_\infty (-\alpha q^{a+c}; q^{2a})_\infty (-\alpha^{-1}q^{a-c}; q^{2a})_\infty$$
$$\text{x} \sum_{m=-\infty}^{\infty} S(\beta, z, q, em)\ \alpha^{dm} q^{-ad^2m^2+cdm}$$
$$+ (q^{2a}; q^{2a})_\infty (-\alpha q^{a+c-b}; q^{2a})_\infty (-\alpha^{-1}q^{a-c+b}; q^{2a})_\infty$$
$$\text{x} \sum_{m=-\infty}^{\infty} S(\beta, z, q, em+1)\ \alpha^{dm} q^{-ad^2m^2+(c-b)dm}$$
$$+ \ldots (q^{2a}; q^{2a})_\infty (-\alpha q^{a+c-eb+b}; q^{2a})_\infty (-\alpha^{-1}q^{a-c+eb-b}; q^{2a})_\infty$$
$$\text{x} \sum_{m=-\infty}^{\infty} S(\beta, z, q, em+e-1)\ \alpha^{dm} q^{-ad^2m^2+(c-eb+b)dm} \tag{2.1}$$

where 2ad – be = O and (d, e) = 1.

Proof : By grouping terms with m ≡ r (mod e), r = O, 1, . . . , e-1 in the left hand side of (2.1), we obtain

$$\sum_{m=-\infty}^{\infty} S(\beta, z, q, m) \sum_{n=-\infty}^{\infty} q^{an^2-bmn+cn}\alpha^n$$

$$= \sum_{m=-\infty}^{\infty} S(\beta, z, q, em)\,\alpha^{dm} q^{-ad^2m^2+cdm} \sum_{n=-\infty}^{\infty} q^{a(n-dm)^2+c(n-dm)}\alpha^{n-dm}$$

$$+ \ldots + \sum_{m=-\infty}^{\infty} S(\beta, z, q, em+e-1)\,\alpha^{dm} q^{-ad^2m^2+(c-be+b)dm}$$

$$\times \sum_{n=-\infty}^{\infty} q^{a(n-dm)^2+(c-be+b)(n-dm)}\alpha^{n-dm}$$

If we now change n to n + dm and use Jacobi's triple product identity, we obtain (2.1).

Lemma 2. for a, b, c, d, e, f arbitrary integers, z,α,β non-zero and |q| < 1,

$$(2.2.)\ \sum_{m=-\infty}^{\infty} q^{am^2+bm}\alpha^m z^{cm} \sum_{n=-\infty}^{\infty} q^{dn^2+en}\beta^n z^{fn}$$

$$= (q^{2a'};q^{2a'})_\infty \sum_{i=O}^{y-1}\sum_{j=O}^{h-1} \alpha^i\beta^j q^{ai^2+bi+dj^2+ej}\ z^{ci+fj}$$

$$\times\ (-\alpha^y\beta^{-x}q^{b'+2ayi-2dxj+a'};q^{2a'})_\infty$$

$$\times\ (-\alpha^{-y}\beta^x q^{b'+2ayi+2dxj+a'};q^{2a'})_\infty$$

$$\times \sum_{n=-\infty}^{\infty} q^{(dh^2-a'g^2)+n^2(eh+2dhj+g(b'-2dxj+2ayi))}(\alpha^{gy}\beta^{h-gx})^n z^{fhn}$$

Where $x = \dfrac{c}{(c,f)}$, $y = \dfrac{f}{(c,f)}$, $\dfrac{dx}{ay^2+dx^2} = \dfrac{g}{h}$ = with (g,h) = 1, a' = ay^2+dx^2 and by-ex = b'.

Proof: By grouping separately terms with m ≡ r (mod y), r = O, 1, . . . , y-1 and changing n to n-xm, the left hand side of (2.2) can be written as a sum of y number of series of the form (2.1). Applying Lemma 1 to each of these y series, we obtain (2.2)
Remark 1. Andrews' result [3, Lemma 3] stated is a particular case of (2.2) and the Laurent expansion of the product can be obtained by applying (2.2) successively.

Lemma 3. For z, $\alpha_1, \ldots, \alpha_k$ non-zero, a_i, b_i, c_i (i = 1, . . . , k) integers and |q| < 1.

$$\sum_{n_1=-\infty}^{\infty} q^{a_1n_1^2+b_1n_1}\ \alpha_1^{n_1} z^{c_1n_1}\ \ldots \tag{1.2.3}$$

$$\sum_{n_k=-\infty}^{\infty} q^{a_kn_k^2+b_knk}\ \alpha_k^{n_k}\alpha^{c_kn_k}$$

$$= \sum_{i_1=O}^{y_1-1} \ldots \sum_{i_{k-1}=O}^{y_{k-1}-1}\sum_{j_i=O}^{h_1-1} \ldots \sum_{j_{l-1}=O}^{h_{k-1}-1} =$$

$$\sum_{n=1}^{k-1}{}_q (\theta_{n-11}i_n^2+\theta_{n-12}i_n{}^{+a}{}_{n+1}j_n^2+b_{n+1}j_n) \sum_{n=1}^{k-1}{}_z (c_nh_{n-1}i_n{}^{+c}{}_{n+1}j_n)\ldots$$

$$\alpha_1^{i_1+g_1y_1i_2+g_1y_1g_2y_2y_3i_3+\ldots+g_1y_1g_2y_2\ldots g_{k-2}y_{k-2}y_{k-1}i_{k-1}}$$

$$\alpha_2^{i_2+(h_1-g_1x_1)i_2+(h_1-g_1x_1)g_2y_2i_3+\ldots+(h_1-g_1x_1)\ldots g_{k-2}y_{k-2}y_{k-2}i_{k-12}}$$

$$\alpha_3^{i_2+(h_2-g_2x_2)i_3+(h_2-g_2x_2)g_3y_3i_4+\ldots+(h_2-g_2x_2)\ldots g_{k-2}y_{k-2}y_{k-2}i_{k-12}}$$

$$\alpha_{k-1}^{j_{k-2}+(h_{k-2}-g_{k-2}x_{k-2})i_{k-1}}\alpha_k^{j_{k-1}}$$

$$\times\ (q^{2a_i};q^{2a_i})_\infty \ldots (q^{2a^1_{k-1}};q^{2a^1_{k-1}})_\infty$$

$$\times(-\alpha_1^{y_1}\alpha_2^{x_1}q^{-b_i+2a_1}y_1i_1+2a_2x_1j_1+a_i;q^{2a_i})_\infty$$

$$\times(-(\alpha_1^{g_1y_1}\alpha_2^{h_1-g_1x_1})^{y_2}\alpha_3^{-x_2}q^{b_2'+2\theta_{11}y_2i_2-2a_3x_2j_2+a_2'};q^{2a_2'})_\infty$$

$$\times(-(\alpha_1^{g_1y_1}\alpha_2^{h_1-g_1x_1})^{g_2y_2}\alpha_3^{(h_2-g_2x_2)g_3y_3}\ldots\alpha_{k-1}^{h_{k-2}-g_{k-2}x_{k-2}})^{-y_{k-1}}$$

$$\times(-(\alpha_1^{g_1y_1}\alpha_2^{h_1-g_1x_1})^{-y_2}\alpha_3^{x_2}q^{-b_2-2\theta_{11}y_2i_2+2a_3x_2j_2+a_2};q^{2a_2})_\infty$$

$$\times(-(\alpha_1^{g_1y_1}\alpha_2^{h_1-g_1x_1})^{g_2y_2}\alpha_3^{(h_2-g_2x_2)g_3y_3}\ldots\alpha_{k-1}^{h_{k-2}-g_{k-2}x_{k-2})-y_{k-1}}$$

$$\alpha_k^{x_{k-1}}q^{b_{k-1}'+2\theta_{1k-2}y_{k-1}i_{k-1}-2a_kx_{k-1}j_{k-1}+a_{k-1}'};q^{2a_{k-1}'})_\infty$$

$$\times(-((\alpha_1^{g_1y_1}\alpha_2^{h_1-g_1x_1})^{g_2y_2}\alpha_3^{(h_2-g_2x_2)g_3y_3}\ldots\alpha_{k-1}^{h_{k-2}-g_{k-2}x_{k-2})^{y_{k-1}}}$$

$$\alpha_k^{-x_{k-1}}q^{-b_{k-1}'-2a_{k-1}i_{k-1}+2a_kx_{k-1}j_{k-1}+a_{k-1}'};q^{2a_{k-1}'})_\infty$$

$$\times\sum_{n=-\infty}^{\infty}(\alpha_1^{g_1\ldots g_{k-1}y_1\ldots y_{k-1}}\alpha_2^{(h_1-g_1x_1)g_2\ldots g_{k-1}y_2\ldots y_{k-1}}$$

$$\ldots\alpha_{k-1}^{(h_{k-2}-g_{k-2}x_{k-2})g_{k-1}y_{k-1}}\alpha_k^{h_{k-1}-g_{k-1}x_{k-1}})^n$$

$$q^{\theta_{k-11}n^2+\theta_{k-12}n}z^{c_kh_{k-1}n}$$

Where $x_n=\dfrac{c_nh_{n-1}}{(c_nh_{n-1},c_{n+1})}$, $y_n=\dfrac{c_{n+1}}{(c_nh_{n-1},c_{n+1})}$, $h_o=1$,

$\Theta_{o1}=O=\Theta_{o2}$, $\dfrac{a_{n+1}x_n}{\theta_{n-11}y_n^2+a_{n+1}x_n^2}=\dfrac{g_n}{h_n}$ with $(g_n, h_n)=1$,

$$a_n'=\theta_{n-11}y_n^2+a_{n+1}x_n^2,\ b_n'=\theta_{n-12}y_n-b_{n+1}x_n,,$$

$\Theta_{n1}=a_{n+1}h_n^2-g_n^2a_n'$ and

$\Theta_{n2}=b_{n+1}h_n+g_n(b_n'+2\Theta_{n-11}y_ni_n-2a_{n+1}x_nj_n)$.

Proof : Applying Lemma 1.2 successively we obtain (1.2.3).

Lemma 4. For a > 0, $a_1, \ldots, a_{k-1}$ integers and $|q| < 1$, the series.

(1.2.4) $$\sum_{n_1,\ldots,n_{k-1}=-\infty}^{\infty}q^{a(\sum_{i=1}^{\infty}n_1^2+\sum_{1\le i<j\le k-1}n_in_j)+\sum_{i=1}^{k-1}a_in_i},$$

can be expressed as a sum of $2^{k-2}\,3^{k-3}\,4^{k-4}\ldots(k-1)^{k-(k-1)}$ infinite products.

Proof: First step. By grouping separately terms with $n_1, \ldots, n_{k-2}$ even and $n_1, \ldots, n_{k-2}$ odd, the series (1.2.4) can be written as the sum of 2^{k-2} series each of which will be of the form

$$q^{m_1}\sum_{n_1,\ldots,n_{k-2}=-\infty}^{\infty}q^{a(3n_1^2+\ldots+3n_{k-2}^2+2\sum_{1\le i<j\le k-2}n_in_j)+\sum_{i=1}^{k-2}b_in_i},$$

$$\text{X} \sum_{n_{k-1}=-\infty}^{\infty} q^{an_{k-1}^2 + .bn_{k-1}},$$

Where m_1, b_1, . . . , b_{k-2}, b are integers. Here the second series can be written as an infinite product by Jacobi's triple product identity. Thus it suffices to express the first series as a product.

Second step: By grouping separately terms with $n_1, \ldots, n_{k-3} \equiv r \pmod 3$, r = 0, 1, 2, the first series of the first step can be written as the sum of 3^{k-3} series each of which will be of the form

$$q^{m_2} \sum_{n_1,\ldots,n_{k-3}=-\infty}^{\infty} q^{a(24n_1^2+\ldots+24n_{k-3}^2+12\sum_{1\le i<j\le k-3} n_i n_j)+\sum_{i=1}^{k-3} c_i n_i},$$

$$\text{x} \sum_{n_{k-2}=-\infty}^{\infty} q^{3an_{k-2}^2 + .cn_{k-2}},$$

Where m_2, c_1, . . . , c_{k-3}, c are all integers.

Proceeding similarly we arrive at the (k-2)-th step namely:

(k-2) – th step. By grouping separately terms with $n_1 \equiv r \pmod{k-1}$, r = 0, 1, . . . , k-2, the first series of the (k-3)-th step can be written as a sum of $(k-1)^{k-(k-1)}$ = k-1 series each of which will be of the form

$$q^{m_{k-2}} \sum_{n_1=-\infty}^{\infty} q^{\alpha_1 n_1^2 + \beta_1 n_1} \sum_{n_2=-\infty}^{\infty} q^{\alpha_2 n_2^2 + \beta_2 n_2},$$

(where m_{k-2}, α_i, β_i (1 = 1,2) are integers) which are explicit infinite products.

Conclusion

From steps 1 to (k-2) it is clear that the series (1.2.4) can be written as a sum of

$$2^{k-2}\, 3^{k-3}\, 4^{k-4} \ldots (k-1)^{k-(k-1)}$$

Infinite products. This proves Lemma 1.4.

Remark 2. By putting a = 1, a_1 = O - . . . = a_{k-1} in Lemma 1.4 and using Theorem 5.2 in [2], we obtain a representation of $C\phi_{k,1}(q) = C\phi_k(q)$ as a sum of infinite products, the number of such products being $2^{k-2}\, 3^{k-3} \ldots (k-1)$.

Remark 3. Multiplying k times and equating the constant terms, we find $C\phi_{k,2}(q)$ to be a sum of ([k/1] + 1) series (the square bracket denoting the integral part) of the form (1.2.4). It follows from Lemma 1.4 that $C\phi_{k,2}(q)$ is a sum of

$$([k/w] + 1)\, 2^{k-2}\, 3^{k-3} \ldots (k-1)$$

infinite products. Similarly, Lemma 1.4 we can obtain a representation of $C\phi_{k,3}(q)$ as a sum of infinite products.

Remark 4. Applying Lemma 1.4 with k = 4 we obtain the following result which can be used to write down the actual expressions of $C\phi_{4,h}(q)$ as sums of explicit infinite products for any h.

For a > O, b, c, d integers and |q| < 1,

$$\sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{a(n_1^2+n_2^2+n_3^2+n_1n_2+n_2n_3+n_3n_1)+bn_1+cn_2+dn_3}$$

$$=(q^{2a};q^{2a})_\infty(q^{6a};q^{6a})_\infty (q^{48a};q^{48a})_\infty \{ (-q^{a+d};q^{2a})_\infty (-q^{-a-d};q^{2a})_\infty$$
$$\text{X}[(-q^{24a+6b-2d-2c}; q^{48a})_\infty (-q^{24a-6b+2d+2c}; q^{48a})_\infty$$
$$\text{X}(-q^{3a+2c-d}; q^{6a})_\infty (-q^{3a-2c+d}; q^{6a})_\infty$$
$$+ q^{3a+2b-d} (-q^{40a+6b-2d-2c}; q^{48a})_\infty (-q^{8a-6b+2d+2c}; q^{48a})_\infty$$
$$\text{X}(-q^{5a+2c-d}; q^{6a})_\infty (-q^{a-2c+d}; q^{6a})_\infty$$
$$+ q^{12a+4b-2d} (-q^{56a+6b-2d-2c}; q^{48a})_\infty (-q^{-8a-6b+2d+2c}; q^{48a})_\infty$$

$x\ (-q^{7a+2c-d}; q^{6a})_\infty \ (-q^{-a-2c+d}; q^{6a})_\infty\ ]$
$+ q^{a+c}\ (-q^{2a+d}; q^{2a})_\infty \ (-q^{-d}; q^{2a})_\infty$
$x\ [(-q^{24a+6b-2c-2d}; q^{48a})_\infty \ (-q^{-24a-6b+2c+2d}; q^{48a})_\infty$
$x\ (-q^{6a+2c-d}; q^{6a})_\infty \ (-q^{-2c+d}; q^{6a})_\infty$
$+ q^{4a+2b-d}\ (-q^{40a+6b-2d-2c}; q^{48a})_\infty \ (-q^{-8a-6b+2d+2c}; q^{48a})_\infty$
$x\ (-q^{8a+2c-d}; q^{6a})_\infty \ (-q^{-2a-2c-d}; q^{6a})_\infty$
$+ q^{14a+4b-2d}\ (-q^{56a+6b-2c-2d}; q^{48a})_\infty \ (-q^{-8a-6b+2c+2d}; q^{48a})_\infty$
$x\ (-q^{10a+2c-d}; q^{6a})_\infty \ (-q^{-4a-2c+d}; q^{6a})_\infty]$
$+ q^{a+b}\ (-q^{2a+d}; q^{2a})_\infty \ (-q^{-d}; q^{2a})_\infty$
$x\ [(-q^{32a+6b-2c-2d}; q^{48a})_\infty \ (-q^{-16a-6b+2c+2d}; q^{48a})_\infty$
$x\ (-q^{4a+2c-d}; q^{6a})_\infty \ (-q^{-2a-2c+d}; q^{6a})_\infty$
$+ q^{6a+2b-d}\ (-q^{48a+6b-2c-2d}; q^{48a})_\infty \ (-q^{-6a-2c+2d}; q^{48a})_\infty$
$x\ (-q^{6a+2c-d}; q^{6a})_\infty \ (-q^{-2c+d}; q^{6a})_\infty$
$+ q^{18a+4b-2d}\ (-q^{64a+6b-2c-2d}; q^{48a})_\infty \ (-q^{-16a-6b+2c+2d}; q^{48a})_\infty$
$x\ (-q^{8a+2c-d}; q^{6a})_\infty \ (-q^{-2c+d}; q^{6a})_\infty]$
$+ q^{3a+b+c}\ (-q^{3a+d}; q^{2a})_\infty \ (-q^{-a-d}; q^{2a})_\infty$
$x\ [(-q^{32a+6b-2c-2d}; q^{48a})_\infty \ (-q^{-16a-6b+2c+2d}; q^{48a})_\infty$
$x\ (-q^{7a+2c-d}; q^{6a})_\infty \ (-q^{-a-2c+d}; q^{6a})_\infty$
$+ q^{7a+2b-d}\ (-q^{48a+6b-2c-2d}; q^{48a})_\infty \ (-q^{-6b+2c+2d}; q^{48a})_\infty$
$x\ (-q^{9a+2c-d}; q^{6a})_\infty \ (-q^{-3a-3c+d}; q^{6a})_\infty$
$+ q^{20a+4b-2d}\ (-q^{64a+6b-2c-2d}; q^{48a})_\infty \ (-q^{-16a-6b+2c+2d}; q^{48a})_\infty$
$x\ (-q^{11a+2c-d}; q^{6a})_\infty \ (-q^{-5a-2c+d}; q^{6a})_\infty]\ \}.$

Representation of $C\phi_{k,h}$ (q): Theorem 1. For arbitrary positive integers k and h, $C\phi_{k,h}$ (q) can be expressed as a sum of infinite products.

Proof: For z, $\alpha_1, \ldots, \alpha_h$ all non – zero and $|q| < 1$, we consider the product

1.3.1. $$(z\alpha_1 q)_\infty\,(z\alpha_2 q)_\infty \ldots (z\alpha_h q)_\infty \quad x\,(z^{-1}\alpha_1^{-1})_\infty\,(z^{-1}\alpha_2^{-1})_\infty \ldots (z^{-1}\alpha_h^{-1})_\infty,$$

Which on using Jacobi's triple product identity can be written as

$$(q)_\infty^{-h} \sum_{n_1=-\infty}^{\infty} (-1)^{n_1} q^{\binom{n_1+1}{2}} \alpha_1^{n_1} z^{n_1}$$
$$\sum_{n_h=-\infty}^{\infty} (-1)^{n_h} q^{\binom{n_h+1}{2}} \alpha_h^{n_h} z^{n_h}$$
. . .

By applying Lemma 1.3 we obtain the Laurent expansion of the product (1.3.1). Substituting $\alpha_1 = \omega_2 = \omega^2, \ldots, \alpha_h = \omega^h$, where $\omega = \exp(2\pi i/h+1)$ in that Laurent expansion, multiplying the resulting identity k times and equating the constant terms, we find $C\phi_{k,h}$ (q) as a sum of series of the form (1.2.4) which by Lemma 1.4 is a sum of $2^{k-2}\ 3^{k-3}\ 4^{k-4} \ldots (k-1)$
Infinite products.

REFERENCES

1. Andrews, G.E. The Theory of Partitions, Encyclopedia of Math. And Its Applications, Vol. 2, Addison – Wesley, Reading, 1976.
2. Andrews, G.E. Generalised Frobenius Partitions, Memoirs of the Amer.Math.Soc. Vol. 49, No. 301 (1984), 1-44.
3. Andrews, G.E. Hecke Modular Forms and the Kac – Peterson Identities, Trans.Amer.Math.Soc.Vol. 283, No.2 (1984), 451-458.
4. Andrews, G.E. q-series, Their Development and Application in Analysis, Number Theory, Combinatories, Physics and Computer Algebra, Amer.Math.Soc. 1986.
5. Bellman, R.A Brief Introduction to Theta Functions, Holt, Rinehart and Winston, New York, 1961.
6. Erdos P. and J. Lehner, The Distribution of the Number of Summands in the Partitions of a Positive Integer, Duke Math.J.8 (1941), 335-345.
7. Euler, L. Introduction on Analysis in Infinitorum, vol. I, Marcum-Michaelem Bousquet, Lausannae, 1748.
8. Frobenius, G. Uber die Charaktere der Symmetrischen Gruppe, Sitzber, Preuss.Akad.Berlin, (1900), 516-534.
9. Gauss, C.F. Werke Vol.2, Konigliche Gesselschaft der Wissenschaften, Gottingen, 1863.
10. Gordon, B. Two New Representations of the Partition Function, Proc.Amer.Math.Soc.13 (1962), 869-873.
11. Gordon B.and L.Houten, Notes on Plane Partitions II, J.Comb.Th.4 (1968), 81-99.

12. Hardy G.H.and S.Ramaujan, Asymptotic Formulae in Combinatory Analysis, Proc.London Math.Soc.(2), 17 (1918), 75-115.
13. Jacobi, C.G.J.Fundamenta Nova Theoriae Funktonum Ellipticarum, Regiomenti Fratrum Borntrager, 1829 (reprinted in Gesammelte Werke, Vol. 1, 49-239, Reimer, Berlin, 1881).
14. Kolitsch, L.W. Some Analytic and Arithmetic Properties of Generalised Frobenium Partitions, Ph.D Thesis, Pennsylvania State University, May 1985.
15. Lehmer, D.H. On the Hardy-Ramanujan Series for the Partition Function, J. London Math.Soc.12 (197), 171-176.
16. Newman, D.J. The Evaluation of the Constant in the Formula for the Number of Partitions of n, Amer.Jour.Math.73 (1951), 599-601.

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