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RESEARCH ARTICLE

ON BIPOLAR-VALUED FUZZY SETS AND THEIR OPERATIONS

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ABSTRACT

In this paper, we present a brief overview on bipolar-valued fuzzy sets which is an extension of fuzzy set theory. A new operations defined over the bipolar-valued fuzzy sets some properties of this operations are discussed and also we introduce the definition of bipolar-valued fuzzy relations various properties like symmetry, reflexivity, transitivity etc. are studied.

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INTRODUCTION

In 1965, Zadeh [6] introduce the notion of a fuzzy subsets of a set, fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. Since then it has become a vigorous area of research indifferent domains, there have been a number of generalization of this fundamental concept such as intuitionist fuzzy set, interval-valued fuzzy set, vague sets, soft set etc. Lee K. M in 2000 introduced the notion of bipolar-valued fuzzy set are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0,1]$ to$[-1,1]$. Ashram Bormann salid [2] in 2009 introduced bipolar-valued fuzzy BCK/BCI-algebra. K. Young Ja Lee [4] in 2009 introduce bipolar-valued fuzzy sub algebra, bipolar-valued fuzzy ideal and some related properties are discussed.

1 Preliminaries

1.1 Definition

Let X be a nonempty set. A fuzzy set A is drawn from X is defined s $A = \{(x, \mu_A(x)): x \in X\}$, where $\mu_A: X \longrightarrow [0,1]$ is the membership function of the fuzzy set A. Fuzzy set is a collection of object with graded membership that is having degree of membership.

Example: The whole concept can be illustrated with this example let's talk about people and "youth fullness "In this case the set S is the set of people. A fuzzy subset young is also defined, which answers the question "To what degree is person x young? To each person in the universe discourse we have to assign a degree of membership in the fuzzy subset young. The easiest way to do this is with membership function based on the person's age

$$\text{YOUNG }(x) = \begin{cases} 1, & if \;\; age(x) \leq 20 \\ \frac{30-x}{10}, & if \;\; 20 < x \leq 30 \\ 0, & if \;\; x > 30 \end{cases}$$

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1.2 Definition [6]

A fuzzy set is empty if and only if its membership function is identically zero on X.

1.3 Definition [6]

Two fuzzy sets A and B are equal, written as $A = B$, *if and only if* $f_A(x) = f_B(x)$ *for all* x *in* X. *(In the of sequel, instead of writing* $f_A(x) = f_B(x)$ *for all* x *in* X, *we shall write more simply* $f_A = f_B$ *.)*

1.4 Definition [6]

The complement of a fuzzy set A is denoted by A' *and is defined by* $f_{A'} = 1 - f_A$.

1.5 Definition [6]

The union of two fuzzy sets A and B with respective membership functions $f_A(x)$ *and* $f_B(x)$ *is a fuzzy set* C, *written as* $C = A \cup B$, *whose membership function is related to those of A and B by* $f_c(x) = Max[f_A(x), f_B(x)],\ x \in X$

1.6 Example

Let $X = \{1,2,3,4,5\}$ and let $A = \{(1,0.3), (2,0.2), (3,0.2), (4,0.8), (5,0.6)\}$ and $B = \{(1,0.2), (2,0.5), (3,0.9), (4,0.7), (5,0.8)\}$. Then

$f_{A\cup B} = \{(1,0.3), (2,0.5), (3,0.9), (4,0.8), (5,0.8)\}$.

Note $\cup$ has the associative property, that is, $A \cup (B \cup C) = (A \cup B) \cup C$.

1.7 Theorem [6]

The union of A and B is the smallest fuzzy set containing both A and B. *More precisely, if D is any fuzzy set which contains both A and* B, *then it also contains the union of A and* B.

Proof: To show that this note is equivalent to (1.3) . We note that C as defined by (1.3) contains both A and B. Since $C \supset A$ implies $f_C \geqq f_A$. Since $C \supset B$ implies $f_C \geqq f_B$. Therefore, we have $Max[f_A(x), f_B(x)] \geqq f_A(x)$ for every $x \in X$ and $Max[f_A(x), f_B(x)] \geqq f_B(x)$ for every $x \in X$. Furthermore, if D is any fuzzy set containing both A and B, then $f_D \geqq f_A$ and $f_D \geqq f_B$. Hence $f_D \geqq f_{A\cup B}$. But $f_{A\cup B} = f_C$. Therefore $f_D(x) \geqq Max[f_A(x), f_B(x)] = f_C(x)$ for every $x \in X$ which implies that $f_C \leqq f_D$. Hence $C \subset D$. Therefore C is the smallest fuzzy set containing both A and B. That is, the union of A and B is the smallest fuzzy set containing both A and B.

1.8 Definition [6]

The intersection of two fuzzy sets A and B with respective membership functions $f_A(x)$ *and* $f_B(x)$ *is a fuzzy set* C, *written as* $C = A \cap B$, *whose membership function is related to those of A and B by* $f_c(x) = Min[f_A(x), f_B(x)]\ ,\ for\ all\ x \in X$

1.9 Example

Let $X = \{1,2,3,4,5\}$ and let $A = \{(1,0.1), (2,0), (3,0.3), (4,0.5), (5,0.6)\}$ and $B = \{(1,0.3), (2,0.5), (3,0.7), (4,0.8), (5,0.4)\}$. Then

$$f_{\boldsymbol{A\cap B}} = \{(1,0.1), (2,0), (3,0.3), (4,0.5), (5,0.4)\}.$$

Note: $\cap$ has the associative property, that is, $A \cap (B \cap C) = (A \cap B) \cap C$.

The intersection of A and B is the largest fuzzy set which is contained in both A and B.

1.10 Definition [6]

The algebraic product of A and B is denoted by AB and is defined in terms of the membership functions of A and B by the relation

$$f_{AB} = f_A\, f_B.$$

Clearly, $AB \subset A \cap B$.

1.11 Definition [6]

The algebraic sum of A and B is denoted by $A + B$ *and is defined by*

$$f_{A+B} = f_A + f_B$$

provided the sum $f_A + f_B$ is less than or equal to unity. Thus, unlike the algebraic product, the algebraic sum is meaningful only when the condition $f_A(x) + f_B \leqq 1$ is satisfied for all x.

1.12 Definition [6]

The absolute difference of A and B is denoted by $|A - B|$ and is defined by $f_{|A-B|} = |f_A - f_B|$. Note that in case of ordinary sets $|A - B|$ reduces to the relative complement of $A \cap B$ in $A \cup B$.

1.13 Definition [6]

The composition of two fuzzy relations A and B is denoted by $B \circ A$ and is defined as a fuzzy relation in X whose membership function is related to those of A and B by

$$f_{B\circ A}(x, y) = \sup_{v} Min[f_A(x, v), \quad f_B(v, y)].$$

Note that the operation of composition has the associative property $A \circ (B \circ C) = (A \circ B) \circ C$.

1.14 Definition [6]

Let T be a mapping from X to a space Y. Let B be a fuzzy set in Y with membership function $f_B(y)$. The inverse mapping of T^{-1} induces a fuzzy set A in X whose membership function is defined by $f_A(x) = f_B(y)$, $y \in Y$ for all x in X which are mapped by T into y.

1.15 Definition [6]

A fuzzy set A is convex if and only if the sets Γ_α defined by

$$\Gamma_\alpha = \{x \mid f_A(x) \geqq \alpha\}$$

are convex for all α in the interval $(0,1]$.

Alternate definition of fuzzy convex: A fuzzy set A is convex if and only if

$$f_A(\lambda x_1 + (1-\lambda)x_2) \geqq Min\big(f_A(x_1), f_A(x_2)\big),$$

for all x_1, x_2 in X and all λ in $[0,1]$.

1.16 Results [6]

To show that the equivalence between the above definitions.

Proof: If A is convex in the sense of the first definition and $\alpha = f_A(x_1) \leqq f_A(x_2)$, then $x_2 \in \Gamma_\alpha$ and $\lambda x_1 + (1-\lambda)x_2 \in \Gamma_\alpha$ by the convexity of Γ_α. Hence

$$f_A(\lambda x_1 + (1-\lambda)x_2) \geqq \alpha = f_A(x_1) = Min\big(f_A(x_1), f_A(x_2)\big).$$

Conversely, if A is convex in the sense of second definition and $\alpha = f_A(x_1)$, then Γ_α may be regarded as the set of all points x_2 for which $f_A(x_2) \geqq f_A(x_1)$. In virtue of (1.24), every point of the form $\lambda x_1 + (1-\lambda)x_2$, $0 \leqq \lambda \leqq 1$, is also in Γ_α and hence Γ_α is convex set.

1.17 Theorem

If A and $B$are convex, so is their intersection.

Proof: Let $C = A \cap B$. Then

$$f_C(\lambda x_1 + (1-\lambda)x_2) = Min\big(f_A(\lambda x_1 + (1-\lambda)x_2), f_B(\lambda x_1 + (1-\lambda)x_2)\big) \qquad (1.25)$$

Since A and $B$are convex, we have

$$f_A(\lambda x_1 + (1-\lambda)x_2) \geqq Min\big(f_A(x_1), f_A(x_2)\big)$$

$$f_B(\lambda x_1 + (1-\lambda)x_2) \geqq Min\big(f_B(x_1), f_B(x_2)\big).$$

Now (1.25) implies that

$$f_C(\lambda x_1 + (1-\lambda)x_2) \geqq Min\left(Min\big(f_A(x_1), f_A(x_2)\big), Min\big(f_B(x_1), f_B(x_2)\big)\right)$$

$$\geqq Min\left(Min\big(f_A(x_1), f_B(x_1)\big), Min\big(f_A(x_2), f_B(x_2)\big)\right).$$

$$\geqq Min\big(f_C(x_1), f_C(x_2)\big).$$

Hence intersection of two fuzzy convex is fuzzy convex.

2 MAIN RESULTS

2.1 Definition

Let X be a non-empty set. An bipolar-valued fuzzy set A in X is an object having the form $A = \{(x, \mu_A^+(x), \mu_A^-(x))/x \in X\}$. *where* $\mu_A^+ : X \longrightarrow [0,1]$ *and* $\mu_A^- : X \longrightarrow [-1,0]$ *are mappings. The positive membership degree* $\mu_A^+(x)$ *denoted the satisfaction degree of an element* x *to the property corresponding to a bipolar-valued fuzzy set* $A = \{(x, \mu_A^+(x), \mu_A^-(x))/x \in X\}$, *and the negative membership degree* $\mu_A^-(x)$ *denotes the satisfaction degree of* x *to some implicit counter-property of* $A = \{(x, \mu_A^+(x), \mu_A^-(x))/x \in X\}$.

2.2 Example

Let $A = \{(a, 0.6, -0.4), (b, 0.8, -0.3), (c, 0.5, -0.5)\}$ is a bipolar-valued fuzzy set of $X = \{a, b, c\}$.

In the canonical representation, it is possible for elements x to be $\mu_A^+(x) \neq 0$ and $\mu_A^-(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain. The reduced representation of a bipolar-valued fuzzy set A on the domain x has the following shape $A = \{(x, \mu_A^R(x))/x \in X\}$, $\mu_A^R : X \longrightarrow [-1,1]$ The membership degree $\mu_A^R(x)$ for the reduced representation can be derived from its canonical representation as follows

$$\mu_A^R(x) = \begin{cases} \mu_A^+(x) & if \ \ \mu_A^-(x) = 0 \\ \mu_A^-(x) & if \ \ \mu_A^+(x) = 0 \\ f\big(\mu_A^+(x), \mu_A^-(x)\big) & otherwish \end{cases}$$

. Here $f\big(\mu_A^+(x), \mu_A^-(x)\big)$ is an aggregation function to merge pair of positive and negative membership values into a value. Such aggregation functions $f\big(\mu_A^+(x), \mu_A^-(x)\big)$ can be defined in various ways.

2.3 Definition

Two bipolar-valued fuzzy set $A = (x, \mu_A^+, \mu_A^-)$ *and* $B = (x, \mu_B^+, \mu_B^-)$ *of a set* X *are equal, written as* $A = B$, *if and only if* $A^+(x) = B^+(x)$ *and* $A^-(x) = B^-(x)$, *for all* $x \in X$.

2.4 Definition

Let $A = (x, \mu_A^+, \mu_A^-)$ *be a bipolar-valued fuzzy set of a set X. The complement of bipolar-valued fuzzy set A is denoted by* $A^c = \{(x, 1 - A^+(x), -1 - A^-(x))/x \in X\}$.

2.5 Definition

Let $A = (x, \mu_A^+, \mu_A^-)$ *and* $B = (x, \mu_B^+, \mu_B^-)$ *be two bipolar-valued fuzzy sets of a set* X. *Then A contained in B (or A is a subset of B) if and only if* $A^+(x) \leq B^+(x)$ *and* $A^-(x) \geq B^-(x)$, *for all* $x \in X$. *In symbols* $A \subset B \Leftrightarrow A^+(x) \leq B^+(x)$ *and* $A^-(x) \geq B^-(x), \forall x \in X$.

2.6 Definition

Let $A = (x, \mu_A^+, \mu_A^-)$ *and* $B = (x, \mu_B^+, \mu_B^-)$ *be two bipolar-valued fuzzy sets of a set* X. *Then union of* A *and* B *is defined as*
$A \cup B = \{(x, \max(A^+(x), B^+(x)), \min(A^-(x), B^-(x)))/x \in X\}$. *The union of two bipolar-valued fuzzy set A and B are defined as follows* $A \cup B = \{(x, \mu_{A\cup B}(x))/x \in X\}$,
where $\mu_{A\cup B}(x) = \left(\mu_{A\cup B}^+(x), \mu_{A\cup B}^-(x)\right) \Longrightarrow \mu_{A\cup B}^+(x) = \max\big(\mu_A^+(x), \mu_B^+(x)\big), \mu_{A\cup B}^-(x) = \min\big(\mu_A^-(x), \mu_B^-(x)\big), \forall x \in X$.

2.7 Definition

Let $A = (A^+, A^-)$ *and* $B = (B^+, B^-)$ *be two bipolar-valued fuzzy set of a set* X. *Then* $A \cap B = \{(x, \min(A^+(x), B^+(x)), \max(A^-(x), B^-(x))/x \in X\}$.

2.8 Theorem

The union of A and B is the smallest bipolar-valued fuzzy set containing both A and B. More precisely if D is any bipolar-valued fuzzy set which contains both A and B then it also contains the union of A and B.

Proof: Let $A = (x, \mu_A^+, \mu_A^-)$ and $B = (x, \mu_B^+, \mu_B^-)$ be two bipolar-valued fuzzy sets of a set X. Take $C = A \cup B = (x, \mu_C^+, \mu_C^-)$ where $\mu_C^+(x) = \max(\mu_A^+(x), \mu_B^+(x))$ and $\mu_C^-(x) = \min(\mu_A^-(x), \mu_B^-(x))$. Since $C \supset A$ implies $\mu_C^+(x) \geqq \mu_A^+(x), \mu_C^-(x) \leqq \mu_A^-(x)$, since $C \supset B$ implies $\mu_C^+(x) \geqq \mu_B^+(x), \mu_C^-(x) \leqq \mu_B^-(x)$. Therefore we have $\max(\mu_A^+(x), \mu_B^+(x)) \geq (x, \mu_A^+(x), \mu_A^-), \min(\mu_A^-(x), \mu_B^-(x)) . \leq (x, \mu_A^+(x), \mu_A^-(x))$ and $\max(\mu_A^+(x), \mu_B^+(x)) \geq (x, \mu_B^+(x), \mu_B^-(x)), \min(\mu_A^-(x), \mu_B^-(x)) \leq (x, \mu_B^+(x), \mu_B^-(x))$, for every $x \in X$. Furthermore if D is any bipolar-valued fuzzy set containing both A and B then $\mu_D^+(x) \geqq \mu_A^+(x)$, $\mu_D^-(x) \leqq \mu_A^-(x)$ and $\mu_D^+(x) \geqq \mu_B^+(x), \mu_D^-(x) \leqq \mu_B^-(x)$ for every $x \in X$ which implies that $\mu_C^+(x) \geqq \mu_D^+(x), \mu_C^-(x) \leqq \mu_D^-(x)$. Hence $C \subset D$. Therefore C is a smallest bipolar-valued fuzzy aet containing both A and B. That is union of A and B is the smallest bipolar-valued fuzzy set containing both A and B.

2.9 Theorem

The intersection of A and B is the bipolar-valued fuzzy sets which is contained in both A and B. More precisely, if D is any bipolar-valued fuzzy set which contained in both A and B, then it also contain the intersection of A and B.

2.10 Definition

Let $\{A_i : i \in J\}$ be an arbitrary family of bipolar-valued fuzzy set in a set X, where $A_i = (x, A_i^+, A_i^-)$. then

1. $\cup A_i = (x, \cup A_i^+, \cap A_i^-)$
2. $\cap A_i = (x, \cap A_i^+, \cup A_i^-)$.

2.11 Definition

Let $A = (x, \mu_A^+, \mu_A^-)$ and $B = (x, \mu_B^+, \mu_B^-)$ be any two bipolar-valued fuzzy set of a set X, respectively. The algebraic product of A and B is denoted by $A \cdot B$ is defined as $A \cdot B = \{(x, \mu_{A\cdot B}^+(x), \mu_{A\cdot B}^-(x))/x \in X\}$, where $\mu_{A\cdot B}^+(x) = \mu_A^+(x) \cdot \mu_B^+(x)$ and $\mu_{A\cdot B}^-(x) = \mu_A^-(x) \cdot \mu_B^-(x)$.

2.12 Definition

Let $A = (x, \mu_A^+, \mu_A^-)$ and $B = (x, \mu_B^+, \mu_B^-)$ be any two bipolar-valued fuzzy set of a set X, respectively. The algebraic sum of A and B is denoted by $A + B$ is defined as $A + B = \{(x, \mu_{A+B}^+(x), \mu_{A+B}^-(x))/x \in X\}$, where $\mu_{A+B}^+(x) = \mu_A^+(x) + \mu_B^+(x) - \mu_A^+(x) \cdot \mu_B^+(x)$ and $\mu_{A+B}^-(x) = \mu_A^-(x) \cdot \mu_B^-(x)$. provided the sum is less than or equal to unity. Thus unlike the algebraic product the algebraic sum is meaningful only when the condition $\mu_A^+(x) + \mu_B^+(x) \leqq 1$ and $\mu_A^-(x) + \mu_B^-(x) \geqq 1$ is satisfied for all x.

2.13 Definition

Let $A = (x, \mu_A^+, \mu_A^-)$ and $B = (x, \mu_B^-, \mu_B^-)$ be any bipolar-valued fuzzy set of a set X respectively Then algebraic difference of A and B is denoted by $|A - B|$ is defined as $|A - B| = \{(x, \mu_{|A-B|}^+(x), \mu_{|A-B|}^-(x))/x \in X\}$, where $\mu_{|A-B|}^+(x) = \mu_A^+(x) - \mu_B^+(x) + \mu_A^+(x) \cdot \mu_B^+(x)$ and $\mu_{|A-B|}^-(x) = \mu_A^-(x) \cdot \mu_B^-(x)$.

2.14 Definition

Let $A = (x, A^+, A^-)$ and $B = (x, B^+, B^-), \Lambda = (x, \Lambda^+, \Lambda^-)$ be arbitrary bipolar-valued fuzzy sets. The convex combination of A, B and Λ is denoted by $(A^+, B^+; \Lambda^+), (A^-, B^-; \Lambda^-)$ and is defined by the relations $(A^+, B^+, \Lambda^+) = \Lambda^+ A^+ + (\Lambda^+)' B^+$ and $(A^-, B^-; \Lambda^-) = \Lambda^- A^- + (\Lambda^-)' B^-$, where $(\Lambda^+)'$ is the complement of Λ^+ and $(\Lambda^-)'$ is the complement of Λ^-. Written out in terms of membership functions

$$\mu_{(A,B;\Lambda)}^+(x) = \mu_\Lambda^+(x)\mu_A^+(x) + [1 - \mu_A^+(x)]\mu_B^+(x),$$
$$x \in X, \mu_{(A,B;\Lambda)}^-(x) = \mu_\Lambda^-(x)\mu_A^-(x) + [-1 - \mu_A^-(x)]\mu_B^-(x), x \in X$$

2.15 Remark

A basic property of the convex combination of A, B and Λ is expressed by $A \cap B \subset (A, B; \Lambda) \subset \mathrm{A} \cup \mathrm{B}$ for all Λ. This property is an immediate consequence of inequalities $\min(\mu_A^+(x)\mu_B^+(x)] \leqq \lambda\mu_\Lambda^+(x) + (1 - \lambda)\mu_B^+(x) \leqq \max\,(\mu_A^+(x), \mu_B^+(x), \forall x \in X$ *and* $\max(\mu_A^-(x)\mu_B^-(x)] \geqq \lambda\mu_\Lambda^-(x) + (1 - \lambda)\mu_B^-(x) \geqq \min\,(\mu_A^-(x), \mu_B^-(x), \forall x \in X$.

2.16 Definition

Let $A_1, \ldots\ldots, A_n$ be a bipolar-valued fuzzy set in $X_1, \ldots\ldots, X_n$ respectively. The Cartesian product $A_1 \times \ldots . \times A_n$ is an bipolar-valued fuzzy set defined by $A_1 \times \ldots . . \times A_n = \left\{ \begin{matrix} [(x_1, \ldots, x_n), \mu^+_{A_1,\ldots,A_n}(x_1, \ldots, x_n), \mu^-_{A_1,\ldots,A_n}(x_1, \ldots, x_n)]: \\ (x_1, \ldots, x_n) \in (X_1, \ldots, X_n) \end{matrix} \right\}$.

2.17 Definition

Bipolar-valued fuzzy set $A = (A^+, A^-)$ is called a convex bipolar-valued fuzzy set if for all $x, y \in X, \lambda \in [0,1], A^+(\lambda x + (1-\lambda)y) \geq A^+(x) \wedge A^+(y)$ and $A^-(\lambda x + (1-\lambda)y) \leq A^-(x) \vee A^-(x)$.

2.18 Theorem

If A and B are convex, so is their intersection

Proof: Let $C = A \cap B$. Then

$\mu^+_C(\lambda x + (1-\lambda)y) = \min(\mu^+_A(\lambda x + (1-\lambda)y), \mu^+_B(\lambda x + (1-\lambda)y))$
$\mu^-_C(\lambda x + (1-\lambda)y) = \max(\mu^-_A(\lambda x + (1-\lambda)y), \mu^-_B(\lambda x + (1-\lambda)y))$. Since A and B are convex we have $\mu^+_A(\lambda x + (1-\lambda)y) \geq \min(\mu^+_A(x), \mu^+_A(y))$, $\mu^-_A(\lambda x + (1-\lambda)y) \leq \max(\mu^-_A(x), \mu^-_A(y))$, $\mu^+_B(\lambda x + (1-\lambda)y) \geq \min(\mu^+_B(x), \mu^+_B(y))$, $\mu^-_B(\lambda x + (1-\lambda)y) \leq \max(\mu^-_B(x), \mu^-_B(y))$. Now we get
$\mu^+_C(\lambda x + (1-\lambda)y) \geq \min\{\min(\mu^+_A(x), \mu^+_A(y)), \min(\mu^+_B(x), \mu^+_B(y))\}$
$\geq \min\{\min(\mu^+_A(x), \mu^+_B(x)), \min(\mu^+_A(x), \mu^+_B(x))\} \qquad \geq \min(\mu^+_C(x), \mu^+_C(y))$.
$\mu^-_C(\lambda x + (1-\lambda)y) \leq \max\{\max(\mu^-_A(x), \mu^-_A(y)), \max(\mu^+_B(x), \mu^-_B(y))\}$
$\leq \max\{\max(\mu^-_A(x), \mu^-_B(x)), \max(\mu^-_A(y), \mu^-_B(y))\} \qquad \leq \max(\mu^-_C(x), \mu^-_C(y))$.

2.19 Definition

Bipolar-valued fuzzy set A is called a concave bipolar-valued fuzzy set if for all $x, y \in X, \lambda \in [0,1], \mu^+_A(\lambda x + (1-\lambda)y) \leq \mu^+_A(x) \vee \mu^+_A(y)$ and $\mu^-_A(\lambda x + (1-\lambda)y) \geq \mu^-_A(x) \wedge \mu^-_A(y)$. It clear that A is convex bipolar-valued fuzzy set if and only if A^c is a concave bipolar-valued sets

2.20 Definition

Let $A = (\mu^+_A, \mu^-_A)$ and $B = (\mu^+_B, \mu^-_B)$ be two bipolar-valued fuzzy set of a set X. Then composition is defined by $A \circ B = \{(t, \mu^+_{A\circ B}(t), \mu^-_{A\circ B}(t))\}$ where

$$\mu^+_{A\circ B} = \begin{cases} \sup_{t \in xy}\{\min(\mu^+_A(x), \mu^+_B(y)\} & if \ \ t \in xy \\ 0 & otherwise \end{cases}$$
$$\mu^-_{A\circ B} = \begin{cases} \inf_{t \in xy}\{(\max(\mu^-_A(x), \mu^-_B(y)\}, & if \ \ t \in xy \\ 0 & otherewise \end{cases}$$

2.21 Definition

Let X and Y be two empty sets and $f: X \longrightarrow Y$ be a mapping. Let $A \in C(X)$ be a collection of all bipolar-valued fuzzy sets. Then the image of A, under f, denoted by $f(A) = (x, \mu^+_{f(A)}, \mu^-_{f(A)})$ is defined by $\mu^+_{f(A)}(y) = \begin{cases} \vee\{(x, \mu^+_A(x); x \in f^{-1}(y))\}, if \ f^{-1}(y) \neq 0 \\ 0 \qquad otherwise \end{cases}$
$\mu^-_{f(A)}(y) = \begin{cases} \wedge\{(x, \mu^-_A(x); x \in f^{-1}(y))\}, if \ f^{-1}(y) \neq 0 \\ 0 \qquad otherwise \end{cases}$

Let $B \in C(Y)$ then the pre image of B, under f, denoted by $f^{-1}(B) = \left(x, \mu^+_{f^{-1}(B)}, \mu^-_{f^{-1}(B)}\right)$, defined by $\mu^+_{f^{-1}(B)}(x) = \mu^+_B(f(x)), \mu^-_{f^{-1}(B)}(x) = \mu^-_B(f(x))$.

2.22 Theorem

Let $A \cup B$ is a concave bipolar-valued fuzzy set when both A and B are concave bipolar-valued fuzzy sets.

Proof: Let $C = A \cup B$, then

$\mu^+_C(\lambda x + (1-\lambda)y) = \max(\mu^+_A(\lambda x + (1-\lambda)y), \mu^+_B(\lambda x + (1-\lambda)y))$.

$\mu_C^-(\lambda x+(1-\lambda)y)=\min\big(\mu_A^-(\lambda x+(1-\lambda)y),\mu_B^-(\lambda x+(1-\lambda)y)\big)$. Since A and B are concave, $\mu_A^+(\lambda x+(1-\lambda)y)\le \max\big(\mu_A^+(x),\mu_A^+(y)\big)$, $\mu_B^+(\lambda x+(1-\lambda)y)\le \max\big(\mu_B^+(x),\mu_B^+(y)\big)$.

$\mu_A^-(\lambda x+(1-\lambda)y)\ge \min\big(\mu_A^-(x),\mu_A^-(y)\big)$,
$\mu_B^-(\lambda x+(1-\lambda)y)\ge \min\big(\mu_B^-(x),\mu_B^-(y)\big)$.

Now we get,

$\mu_C^+(\lambda x+(1-\lambda)y)\le \max\{\max\big(\mu_A^+(x),\mu_A^+(y)\big),\max\big(\mu_B^+(x),\mu_B^+(y)\big)\}$
$\le \max\{\max\big(\mu_A^+(x),\mu_B^+(x)\big),\max\big(\mu_A^+(y),\mu_B^+(y)\big)\}$
$\le \max\big(\mu_C^+(x),\mu_C^+(y)\big)$.
$\mu_C^-(\lambda x+(1-\lambda)y)\ge \min\{\min\big(\mu_A^-(x),\mu_A^-(y)\big),\min\big(\mu_B^-(x),\mu_B^+(y)\big)\}$
$\ge \min\{\min\big(\mu_A^-(x),\mu_B^-(x)\big),\min\big(\mu_A^-(y),\mu_B^-(y)\big)\}$.
$\ge \min\big(\mu_C^-(x),\mu_C^-(y)\big)$.

Let X,Y,Z and U be ordinary finite non-empty sets. Let U given by the membership functions μ_A^+ and μ_B^+ respectively and the non-membership functions μ_A^- and μ_B^- respectively where $\mu_A^+,\mu_B^+,\mu_A^-,\mu_B^-:U\longrightarrow[0,1]$. $A\times B$ is bipolar-valued set in $U\times U$ defined by $\mu_{A\times B}^+(x,y)=\min\big(\mu_A^+(x),\mu_B^+(y)\big)$, $\mu_{A\times B}^-(x,y)=\max\big(\mu_A^-(x),\mu_B^-(y)\big)$, for all $x,y\in U$.

2.23 Definition

Let $R\subseteq A\times B$, that is $\mu_R^+(x,y)\le\mu_{A\times B}^+(x,y)$ and $\mu_R^-(x,y)\ge\mu_{A\times B}^-(x,y)$ with the condition that $0\le\mu_R^+(x,y)+\mu_R^-(x,y)\le 1$. Then R is an bipolar-valued fuzzy relation from A to B.

2.24 Definition

Given a binary bipolar-valued fuzzy relation between X and Y we can define R^{-1} between Y and X be means of $\mu_{R^{-1}}^+(y,x)=\mu_R^+(x,y),\mu_{R^{-1}}^-(y,x)=\mu_R^-(x,y),\forall(x,y)\in X\times Y$ to which we will call inverse relation of R.

2.25 Definition

Let R and P be two bipolar-valued fuzzy relations between X and Y for every $(x,y)\in X\times Y$ we can define

a. $R\le P\Leftrightarrow\mu_R^+(x,y)\le\mu_P^+(x,y)$ and $\mu_R^-(x,y)\ge\mu_P^-(x,y)$.
b. $R\lesssim P\Leftrightarrow\mu_R^+(x,y)\le\mu_P^+(x,y)$ and $\mu_R^-(x,y)\le\mu_P^-(x,y)$.
c. $R\vee P=\{\big((x,y),\mu_R^+(x,y)\vee\mu_P^+(x,y),\mu_R^-(x,y)\wedge\mu_P^-(x,y)\big)\}$.
d. $R\wedge P=\{\big((x,y),\mu_R^+(x,y)\wedge\mu_P^+(x,y),\mu_R^-(x,y)\vee\mu_P^-(x,y)\big)\}$.
e. $R_c=\{\big((x,y),\mu_R^-(x,y),\mu_R^+(x,y)\big),x\in X,y\in Y\}$.

2.26 Definition

An bipolar-valued fuzzy relation $R=\{(x,y),\mu_A^+(x,y),\mu_A^-(x,y)/x,y\in A\times A\}$. is said to be reflexive if $\mu_A^+(x,x)=1$ and $\mu_A^-(x,x)=0$ for all $x\in X$. Also R is said to be symmetric if $\mu_A^+(x,y)=\mu_A^+(y,x)$ and $\mu_A^-(x,y)=\mu_A^-(y,x)$, fro all $x,y\in A$.

2.27 Theorem

If R is symmetric then so is R^{-1}.

Proof: $\mu_{R^{-1}}^+(x,y)=\mu_R^+(y,x)=\mu_R^+(x,y)=\mu_{R^{-1}}^+(y,x)$
$\mu_{R^{-1}}^-(x,y)=\mu_R^-(y,x)=\mu_R^-(x,y)=\mu_{R^{-1}}^-(y,x),\forall x,y\in U$.

2.28 Theorem

R is symmetric if and only if $R=R^{-1}$.

Proof: Let R be symmetric then

$\mu_{R^{-1}}^+(x,y)=\mu_R^+(y,x)=\mu_R^+(x,y)$
$\mu_{R^{-1}}^-(x,y)=\mu_R^-(y,x)=\mu_R^-(x,y)$ for all $x,y\in U$. So, $R^{-1}=R$.
Conversely, let $R^{-1}=R$

$\mu_R^+(x,y) = \mu_{R^{-1}}^+(x,y) = \mu_R^+(y,x), \mu_R^-(x,y) = \mu_{R^{-1}}^-(x,y) = \mu_R^-(y,x).$

2.29 Definition

If $R_1 = \{(x,y), \mu_1^+(x,y), \mu_1^-(x,y)/x, y \in A \times A\}$ *and* $R_2 = \{(x,y), \mu_2^+(x,y), \mu_2^-(x,y)/x, y \in A \times A\}$. *be a two bipolar-valued fuzzy relations on A then composition denoted by* $R_1 \circ R_2$ *is defined by* $R_1 \circ R_2 = \{((x,y), (\mu_1^+ \circ \mu_2^+)(x,y), (\mu_1^- \circ \mu_2^-)(x,y))/x, y \in A \times A\}$,*where* $(\mu_1^+ \circ \mu_2^+)(x,y) = \sup_{z\in A}\{\min(\mu_1^+(x,z), \mu_2^+(z,y))\}$ *and* $(\mu_1^- \circ \mu_2^-)(x,y) = \inf_{z\in A}\{\max(\mu_1^-(x,z), \mu_2^-(z,y))\}$.

2.30 Definition

An bipolar-valued fuzzy relation R on A is called transitive if $R \circ R \subseteq R$.

2.31 Theorem

If R is a transitive relation then so R^{-1}.

Proof: $\mu_{R^{-1}}^+(x,y) \geq \mu_{R^{-1}\circ R^{-1}}^+(x,y)$.
$\mu_{R^{-1}}^-(x,y) = \mu_R^-(y,x) \leq \mu_{R\circ R}^-(y,x) = \min_{z\in U}[\min(\mu_R^-(y,x), \mu_R^-(z,x))]$
$= \min_{z\in U}\left[\max\left(\mu_{R^{-1}}^-(x,z), \mu_{R^{-1}}^-(z,y)\right)\right] = \mu_{R^{-1}\circ R^{-1}}^-(x,y)$. So $R^{-1} \circ R^{-1} \subseteq R^{-1}$.

2.32 Definition

An bipolar-valued fuzzy relation R on A is called an bipolar-valued fuzzy equivalence relation if R is reflexive, symmetric and transitive.

2.33 Definition

For any bipolar-valued fuzzy set $A = (x, \mu_A^+(x), \mu_A^-(x))$ *of a set X we defined a* (α,β)*-cut of A as the crisp subset* $\{x \in X/\mu_A^+(x) \geq \alpha, \mu_A^-(x) \leq \beta\}$ *of X and it is denoted by* $C_{\alpha,\beta}(A)$.

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