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ASIAN JOURNAL OF
SCIENCE AND TECHNOLOGY

ISSN: 0976-3376

*Asian Journal of Science and Technology*
*Vol. 09, Issue, 05, pp.8099-8106, May, 2018*

# RESEARCH ARTICLE

## ON THE BI-QUADRATIC EQUATION WITH THREE UNKNOWNS $xy+6(x+y)+4(x^2+y^2)+4=176z^4$

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**ARTICLE INFO**

***Article History:***
Received 17th February, 2018
Received in revised form
20th March, 2018
Accepted 06th April, 2018
Published online 30th May, 2018

***Key words:***

Bi-quadratic,
Diophantine

**ABSTRACT**

Stu The bi-quadratic non-homogeneous equation with three unknowns represented by the Diophantine equation $xy+6(x+y)+4(x^2+y^2)+4=176z^4$ is analyzed for its patterns of non –zero distinct integral solutions.



## INTRODUCTION

The bi-quadratic Diophantine (homogeneous or non-homogeneous) equations offer an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-12] for ternary non-homogeneous bi-quadratic equations .This communication concerns with an interesting bi-quadratic non-homogeneous equation with three unknowns represented by $xy+6(x+y)+4(x^2+y^2)+4=176z^4$ for determining its infinitely many non-zero integral solutions.

### Method of Analysis

The equation under consideration is

$$xy+6(x+y)+4(x^2+y^2)+4=176z^4 \tag{1}$$

Assume that $x=u+v \quad and \quad y=u-v$ (2)

where $u\neq v \ and \ u,v\neq 0$

Therefore, (1) becomes

$$U^2+7v^2=176z^4 \tag{3}$$

Where

$$U=3u+2 \tag{4}$$

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Five different patterns of solving (3) are illustrated below

**Pattern: 1**

Assume that

$$z = a^2 + 7b^2 \tag{5}$$

write $176$ as

$$176 = (1 + i5\sqrt{7})(1 - i5\sqrt{7})$$

Applying the method of factorization in (3), we obtain

$$(U + i\sqrt{7}v)(U - i\sqrt{7}v) = (1 + i5\sqrt{7})(1 - i5\sqrt{7})(a + i\sqrt{7}b)^4 (a - i\sqrt{7}b)^4$$

Equating the positive parts on both sides, we get

$$(U + i\sqrt{7}v) = (1 + i5\sqrt{7})(a + i\sqrt{7}b)^4 \tag{6}$$

Expanding the right hand side of (6) and equating the real and imaginary parts, we note that

$$U = a^4 - 140a^3b - 42a^2b^2 + 980ab^3 + 49b^4 \tag{7}$$

$$v = 5a^4 + 4a^3b - 210a^2b^2 - 28ab^3 + 245b^4$$

Comparing (4) and (7), we find that

$$u = \frac{1}{3}\left(a^4 - 140a^3b - 42a^2b^2 + 980ab^3 + 49b^4 - 2\right)$$

Since, our intension is to find integer solutions of the equation (1), we observe that $u$ is an integer for the following choice of $a$ $and$ $b$

$$a = 3A - 1 \text{ and } b = 3B - 1$$

Thus, the value of $u, v$ $and$ $z$ are given by

$$u = 27A^4 + 1224A^3 - 1368A^2 - 480A + 26460AB^3 - 25704AB^2 + 7056AB - 2912B$$
$$+ 9576B^2 - 10584B^3 + 1323B^4 + 4536A^2B - 3780A^3B - 1134A^2B^2 + 282$$

$$v = 405A^4 - 648A^3 - 1512A^2 + 1248A - 2268AB^3 + 13608AB^2 - 8208AB - 1440B$$
$$+ 10584B^2 - 25704B^3 + 19845B^4 + 11016A^2B + 324A^3B - 17010A^2B^2 + 16$$

$$z = 9A^2 - 6A + 63B^2 - 42B + 8$$
(8)

Substituting the values of $u$ $and$ $v$ in (2), the infinitely many non-zero integral solutions to (1) are exhibited by

$$x = 432A^4 + 576A^3 - 2880A^2 + 768A + 24192AB^3 - 12096AB^2 - 1152AB - 4352B$$
$$+ 20160B^2 - 36288B^3 + 21168B^4 + 15552A^2B - 3456A^3B - 18144A^2B^2 + 298$$
$$y = -378A^4 + 1872A^3 + 144A^2 - 1728A + 28728AB^3 - 39312AB^2 + 15264AB - 1472B$$
$$-1008B^2 + 15120B^3 - 18522B^4 - 6480A^2B - 4104A^3B + 15876A^2B^2 + 266$$

and (8)

**Pattern: 2**

Rewrite $176$ as

$$176 = \left(8 + i4\sqrt{7}\right)\left(8 - i4\sqrt{7}\right)$$

Applying the same procedure as explained in pattern 1, we get

$$u = \frac{1}{3}\left(8a^4 - 112a^3b - 336a^2b^2 + 784ab^3 + 392b^4 - 2\right)$$

$$v = \left(4a^4 + 32a^3b - 168a^2b^2 - 224ab^3 + 196b^4\right)$$

We examine that, the value of $u$ is an integer for the following choices of $a \ and \ b$

$$a = 3A \ and \ \ b = 3B - 1$$

Therefore, we get

$$u = 216A^4 + 1008A^3 - 1008A^2 - 784A + 21168AB^3 - 21168AB^2 + 7056AB - 1568B$$
$$+ 7056B^2 - 14112B^3 + 10584B^4 + 6048A^2B - 3024A^3B - 9072A^2B^2 + 130$$

$$v = 324A^4 - 864A^3 - 1512A^2 + 672A - 18144AB^3 + 18144AB^2 - 6048AB - 2352B$$
$$+ 10584B^2 - 21168B^3 + 15876B^4 + 9072A^2B + 2592A^3B - 13608A^2B^2 + 196$$

$$z = 9A^2 + 63B^2 - 42B + 7 \qquad (9)$$

In view of (2), the non-trivial integral solutions to (1) are expressed by

$$x = 540A^4 + 144A^3 - 2520A^2 - 112A + 3024AB^3 - 3024AB^2 + 1008AB - 3920B$$
$$+ 17640B^2 - 35280B^3 + 26460B^4 + 15120A^2B - 432A^3B - 22680A^2B^2 + 326$$
$$y = -108A^4 + 1872A^3 + 504A^2 - 1456A + 39312AB^3 - 39312AB^2 + 13104AB + 784B$$
$$-3528B^2 + 7056B^3 - 5292B^4 - 3024A^2B - 5616A^3B + 4536A^2B^2 - 66$$

and (9)

**Pattern: 3**

We write $176$ as

$$176 = \left(13 + i\sqrt{7}\right)\left(13 - i\sqrt{7}\right)$$

Applying the same procedure as explained in pattern 1, we obtain

$$u=\frac{1}{3}\left(13a^4-28a^3b-546a^2b^2+196ab^3+637b^4-2\right)$$

Since our aim is to evaluate integer solutions to (1), we note that the value of $u$ is an integer when

$$a=3A-1 \ and \ \ b=3B-1$$

Thus, we acquire that

$$u=351A^4-216A^3-1656A^2+928A+5292AB^3+4536AB^2-5040AB-2016B$$
$$+11592B^2-24696B^3+17199B^4+10584A^2B-756A^3B-14742A^2B^2+90$$

$$v=81A^4-1512A^3+1080A^2+864A-29484AB^3+31752AB^2-9936AB+2784B$$
$$-7560B^2+4536B^3+3969B^4-1944A^2B+4212A^3B-3402A^2B^2-304$$

$$z=9A^2-6A+63B^2-42B+8$$
(10)

Substituting the values of $u \ and \ v$ in (2), we search out the integral solutions to (1) are pointed out by

$$x=432A^4-1728A^3-576A^2+1792A-24192AB^3+36288AB^2-14976AB+768B$$
$$+4032B^2-20160B^3+21168B^4+8640A^2B+3456A^3B-18144A^2B^2-214$$
$$y=270A^4+1296A^3-2736A^2+64A+34776AB^3-27216AB^2+4896AB-4800B$$
$$+19152B^2-29232B^3+13230B^4+12528A^2B-4968A^3B-11340A^2B^2+394$$

and (10)

**Pattern: 4**

Equation (4) can be written as

$$U^2+7v^2=64z^4+7\cdot 16z^4$$

$$\Rightarrow U^2-64z^4=7\left(16z^4-v^2\right)$$

$$\Rightarrow \left(U+8z^2\right)\left(U-8z^2\right)=7\left(4z^2+v\right)\left(4z^2-v\right)$$

The above equation is written in the form as follows

$$\frac{\left(U+8z^2\right)}{\left(4z^2+v\right)}=\frac{7\left(4z^2-v\right)}{\left(U-8z^2\right)}=\frac{\alpha}{\beta},(\beta\neq 0)$$

Equating the first and third terms in the above equation, we find that

$$\beta U+z^2(8\beta-4\alpha)-\alpha v=0 \tag{11}$$

Similarly, by equating the second and third terms, we get

$$-\alpha U+z^2(8\alpha+28\beta)-7\beta v=0 \tag{12}$$

Solving (11) and (12) by method of cross multiplication and simplifying, we have

$$z^2 = \alpha^2 + 7\beta^2 \tag{13}$$

$$U = 8\alpha^2 + 56\alpha\beta - 56\beta^2 \tag{14}$$

$$v = -4\alpha^2 + 16\alpha\beta + 28\beta^2 \tag{15}$$

Using (4) and (14), we discover that

$$u = \frac{1}{3}\left(8\alpha^2 + 56\alpha\beta - 56\beta^2 - 2\right) \tag{16}$$

Consider the general solution to the standard equation (13) as

$$\alpha = 7r^2 - s^2$$

$$\beta = 2rs$$

$$z = 7r^2 + s^2$$

Substituting the values of $\alpha$ *and* $\beta$ in (15) and (16), we have

$$u = \frac{1}{3}\left[392r^4 + 8s^4 - 336r^2s^2 + 784r^3s - 112rs^3 - 2\right]$$

$$v = \left[-196r^4 - 4s^4 + 168r^2s^2 + 224r^3s - 32rs^3\right]$$

we discover that the values of $u$ is an integer when

$$r = 3R - 1 \text{ and } s = 3S$$

Then, we have

$$\begin{aligned} u = {} & 10584R^4 - 14112R^3 + 7056R^2 - 1568R + 216S^4 + 1008S^3 - 1008S^2 - 784S \\ & - 9072R^2S^2 - 21168R^2S + 6048RS^2 + 7056RS + 21168R^3S - 3024RS^3 + 130 \end{aligned}$$

$$\begin{aligned} v = {} & -15876R^4 + 21168R^3 - 10584R^2 + 2352R - 324S^4 + 864S^3 + 1512S^2 - 672S \\ & + 13608R^2S^2 - 18144R^2S - 9072RS^2 + 6048RS + 18144R^3S - 2592RS^3 - 196 \end{aligned}$$

$$z = 63R^2 - 42R + 9S^2 + 7 \tag{17}$$

Substituting the values of $u$ *and* $v$ in (2), the two parametric integral solution to (1) are exhibited by

$$\begin{aligned} x = {} & -5292R^4 + 7056R^3 - 3528R^2 + 784R - 108S^4 + 1872S^3 + 504S^2 - 1456S \\ & + 4536R^2S^2 - 39312SR^2 - 3024RS^2 + 13104RS + 39312R^3S - 5616RS^3 - 66 \end{aligned}$$

$$\begin{aligned} y = {} & 26460R^4 - 35280R^3 + 17640R^2 - 3920R + 540S^4 + 144S^3 - 2520S^2 - 112S \\ & - 22680R^2S^2 - 3024SR^2 + 15120RS^2 + 1008RS + 3024R^3S - 432RS^3 + 326 \end{aligned}$$

and (17)

**Pattern: 5**

Rewrite (4) as

$$U^2 + 7v^2 = 169z^4 + 7z^4$$

$$\Rightarrow U^2 - 169z^4 = 7\left(z^4 - v^2\right)$$

$$\Rightarrow \left(U + 13z^2\right)\left(U - 13z^2\right) = 7\left(z^2 + v\right)\left(z^2 - v\right)$$

This equation is written in the form of ratio as

$$\frac{\left(U + 13z^2\right)}{\left(z^2 + v\right)} = \frac{7\left(z^2 - v\right)}{\left(U - 13z^2\right)} = \frac{\alpha}{\beta}, (\beta \neq 0)$$

which is equivalent to the system of double equations

$$\frac{\left(U + 13z^2\right)}{\left(z^2 + v\right)} = \frac{\alpha}{\beta}, (\beta \neq 0)$$

$$\Rightarrow \beta U + z^2\left(13\beta - \alpha\right) - \alpha v = 0 \quad (18)$$

And

$$\frac{7\left(z^2 - v\right)}{\left(U - 13z^2\right)} = \frac{\alpha}{\beta}, (\beta \neq 0)$$

$$\Rightarrow -\alpha U + z^2\left(13\alpha + 7\beta\right) - 7\beta v = 0 \quad (19)$$

Solving (18) and (19) by method of cross multiplication and simplifying, we have

$$z^2 = \alpha^2 + 7\beta^2 \quad (20)$$

$$U = 13\alpha^2 + 14\alpha\beta - 91\beta^2 \quad (21)$$

$$v = -\alpha^2 + 26\alpha\beta + 7\beta^2 \quad (22)$$

In view of (4) and (21), we get

$$u = \frac{1}{3}\left(13\alpha^2 + 14\alpha\beta - 91\beta^2 - 2\right) \quad (23)$$

Employing the general solutions to (20), we note that

$$\alpha = 7r^2 - s^2 \quad (24)$$

$$\beta = 2rs \quad (25)$$

$$z = 7r^2 + s^2 \quad (26)$$

using (24) and (25) in (22) and (23), we get

$$u = \frac{1}{3}\left[637r^4 + 13s^4 - 546r^2s^2 + 196r^3s - 28rs^3 - 2\right]$$

$$v = -49r^4 + s^4 + 42r^2s^2 + 364r^3s - 52rs^3$$

We scrutinize that, the values of $u \; and \; v$ are integers when

$$r = 3R - 1 \; and \; s = 3S - 1$$

On substituting these values, we get the values of $u, v \; and \; z$ as follows

$$u = 17199R^4 - 24696R^3 + 11592R^2 - 2016R + 351S^4 - 216S^3 - 1656S^2 + 928S$$
$$- 14742R^2S^2 + 4536SR^2 + 10584RS^2 - 5040RS + 5292R^3S - 756RS^3 + 90$$

$$v = -3969R^4 - 4536R^3 + 7560R^2 - 2784R - 81S^4 + 15`12S^3 - 1080S^2 - 864S$$
$$+ 3402R^2S^2 - 31752R^2S + 1944RS^2 + 9936RS + 29484R^3S - 4212RS^3 + 304$$

$$z = 63R^2 - 42R + 9S^2 - 6s + 8 \quad (27)$$

Substituting the values of $u \; and \; v$ in (2), the infinitely may non-zero integer solutions to (1) are given by

$$x = 13230R^4 - 29232R^3 + 19152R^2 - 4800R + 270S^4 + 1296S^3 - 2736S^2 + 64S$$
$$- 11340R^2S^2 - 27216R^2S + 12528RS^2 + 4896RS + 34776R^3S - 4968RS^3 + 394$$
$$y = 21168R^4 - 20160R^3 + 4032R^2 + 768R + 432S^4 - 1728S^3 - 576S^2 + 1792S$$
$$- 18144R^2S^2 + 36288R^2S + 8640RS^2 - 14976RS - 24192R^3S + 3456RS^3 - 214$$

and (27)

## Conclusion

In this paper, we find various patterns of non-zero integral solutions to the ternary bi-quadratic equation. In this manner, one can evaluate different pattern of non-zero integral solutions to higher degree Diophantine equation with more than three unknowns.

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