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RESEARCH ARTICLE

ON TWO WEIGHT CRITERIONS FOR THE HARDY LITTLEWOOD MAXIMAL OPERATOR IN BFS

***Lutfi AKIN**

Department of Business Administration, Faculty of Economics and Administrative Sciences, Mardin Artuklu University, Mardin, Turkey

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ABSTRACT

Our aim of this paper is to prove two-weight criterions for the Hardy-Littlewood maximal operator from weighted Lebesgue spaces into Banach function spaces (BFS). We used boundedness of geometric mean operator and sufficient condition on the weights for boundedness of certain sublinear operator from weighted Lebesgue spaces into weighted Musielak-Orlicz spaces

INTRODUCTION

Operator theory were worked by very mathematicien (Akin and Zeren, 2017). Compactification of weighted Hardyoperator in variable exponent Lebesgue spaces has been proven, in (Mamedov *et al.,* 2017). The goal of this investigations were closely connected with the found of criterion for validity of boundedness of Hardy-Littlewood maximal operator in BFS. On a two-weight criteria for multidimensional Hardytype operator in p-convex Banach function spaces and some application has been proven, in (Bandaliev, 2012). In (Cochranand, 1984) was proved the boundedness of Hardyoperator in Orliczspaces. Also, in (Lomakina and Stepanov, 1998) the compactness and measure of non-compactness of Hardytype operator in Banach function spaces was proved. Wereferethenotion of BFS was introduced in (Luxemburg, 1955). Inthisarticle, weestablish an integral-type necessary and sufficient condition on weights which provides the boundedness of the Hardy-Littlewood maximal perator from weighted Lebesgue spaces into p-convex weighted BFS.

Auxillary Statements and Assertions

Let(Ω,μ) be a complete σ-finite measure space. By $L_0=L_0(\Omega,\mu)$ we denote the collection of all real-valued μ-measurable functions on Ω.

Definition.1. (Quinsheng, 1993) Let$w$ be a weight on R^n. The weighted Hardy-Littlewood maximal operator $\mathcal{M}_w$ is defined by

$$\mathcal{M}_w f(x)=\sup_{B:ball,x\in B}\frac{1}{w(B)}\int_B |f(y)|w(y)dy.$$

In the case that $w\equiv 1$, $\mathcal{M}_w$ is the usual Hardy-Littlewood maximal operator$\mathcal{M}$, namely$\mathcal{M}f(x)=\sup_{B:ball,x\in B}\frac{1}{|B|}\int_B |f(y)|dy$.
Given a BFS,Y we can consider its associate space Y' consisting of those $g\in L_0$ that $f.g\in L_1$ for every $f\in Y$ with the usual order and the norm

$$\|g\|_{Y'}=\sup\{\|f.g\|_{L_1}:\|g\|_{Y'}\le 1\}.$$

Note that Y' is a BFS in $(\Omega,\mu)$and a closed norming subspaces. Let$Y$ be a BFS and w be a weight, that is, positive Lebesgue measurable and a.e. finite functions on Ω .

Corresponding author: **Lutfi AKIN**
Department of Business Administration, Faculty of Economics and Administrative Sciences, Mardin Artuklu University, Mardin, Turkey.

Let $Y_w = \{f \in L_0 : fw \in Y\}$. This space is a weighted BFS equipped with the norm $\|f\|_{Y_w} = \|fw\|_Y$(see[8,9]).

Definition 2. (Schep, 1995) Let$Y$ is a BFS. Then Y is called p-convex for $1 \le p \le \infty$ if there exists a constant $M > 0$ such that for all $f_1, \dots, f_n \in Y$

$$\left\| \left(\sum_{k=1}^{n} |f_k|^p\right)^{1/p} \right\|_Y \le M\left(\sum_{k=1}^{n} \|f_k\|_Y^p\right)^{1/p} \text{if } 1 \le p < \infty, \text{or}$$

$$\|\sup_{1\le k\le n} |f_k|\|_Y \le M \max_{1\le k\le n} \|f_k\|_Y \quad \text{if} \quad p = \infty.$$

Similarly Y is called p-concave $1 \le p \le \infty$ if there exists a constant $M > 0$ such that for all $f_1, \dots, f_n \in Y$

$$\left(\sum_{k=1}^{n} \|f_k\|_Y^p\right)^{1/p} \le M \left\| \left(\sum_{k=1}^{n} |f_k|^p\right)^{1/p} \right\|_Y \text{if} \quad 1 \le p < \infty, \text{or } \max_{1\le k\le n} \|f_k\|_Y \le M \|\sup_{1\le k\le n} |f_k|\|_Y \text{if} p = \infty.$$

Lemma 3. [11] Let$1 \le p \le q(x) \le q^+ < \infty$ for all $y \in \Omega_2 \subset R^m$. Then the inequality

$$\left\| \|f\|_{L_p(\Omega_1)} \right\|_{L_{q(.)}(\Omega_2)} \le C_{p,q} \left\| \|f\|_{L_q(\Omega_2)} \right\|_{L_p(\Omega_1)}$$

is valid, where$C_{p,q} = \left(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty + p\left(\frac{1}{\underline{q}} - \frac{1}{\bar{q}}\right) \right) \left(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty \right),$

$q^- = ess\ \inf_{\Omega_2} q(x)$, $q^+ = ess\ \sup_{\Omega_2} q(x)$, $\Delta_1 = \{(x,y) \in \Omega_1 \times \Omega_2 : q(y) = p\}$, $\Delta_2 = \Omega_1 \times \Omega_2 \setminus \Delta_1$ and $f: \Omega_1 \times \Omega_2 \to R$is any measurable function such that

$$\left\| \|f\|_{L_p(\Omega_1)} \right\|_{L_{q(.)}(\Omega_2)} = inf\left\{ \mu > 0 : \int_{\Omega_2} \left(\frac{\|f(.,y)\|_{L_p(\Omega_1)}}{\mu} \right)^{q(y)} dy \le 1 \right\} < \infty \text{ and } \|f(.,y)\|_{L_p(\Omega_1)} = \left(\int_{\Omega_1} |f(x,y)|^p dx \right)^{1/p}.$$

Definition 4. (Musielak, 1983; Diening *et al.*, 2011) Let us$\Omega \in R^n$ a Lebesgue measurable set. A real function $\tau: \Omega \times [0,\infty) \to [0,\infty)$ is called a generalized τ-functionif it satisfies:

a) $\tau(x,.)$ is a τ -function for all $x \in \Omega, \tau(x,.): [0,\infty) \to [0,\infty)$isconvexandsatisfies$\tau(x,0) = 0$, $\lim_{t\to 0^+} \tau(x,t) = 0$
b) $\psi: x \to \tau(x,t)$is measurableforall$t \ge 0$.

Definition 5. (Musielak, 1983Dieningand and Samko, 2007) Let$\tau \in \Phi$ and be ρ_τ defined by$\rho_\tau(f) = \int_\Omega \tau(y, |f(y)|)dy$,forall$f \in L_0(\Omega)$.We put $L_\tau = \{f \in L_0(\Omega): \rho_\tau(\lambda_0 f) < \infty \ for\ some\ \lambda_0 > 0\}and\|f\|_{L_\tau} = inf\left\{\lambda > 0 : \rho_\tau\left(\frac{f}{\lambda}\right) \le 1\right\}$.

The space L_τ is called Musielak-Orliczspace. Let w be a weight function on Ω, i.e.,w is a non-negative, almost everywhere positive function on Ω. In this work we considered the weighted Musielak-Orlicz spaces. We denote$L_{\tau,w} = \{f \in L_0(\Omega): fw \in L_\tau\}$. It is obvious that the norm in this spaces is given by $\|f\|_{L_{\tau,w}} = \|fw\|_{L_\tau}$

Lemma 6. (Bandaliev, 2012) Let$\Omega_1 \subset R^n$ and $\Omega_2 \subset R^m$. Let $(x,t) \in \Omega_1 \times [0,\infty)$, and $\tau(x, t^{1/p}) \in \Phi$ for some $1 \le p < \infty$. Suppose $f: \Omega_1 \times \Omega_2 \to R$. Then

$$\left\| \|f(x,.)\|_{L_p(\Omega_2)} \right\|_{L_\tau} \le 2^{1/p} \left\| \|f(.,y)\|_{L_\tau} \right\|_{L_p(\Omega_2)} \text{is valid.}$$

Definition 7. We say that$\tau \in \Phi$ satisfies the Δ_2-conditionifthereexists$K \ge 2$ such that $\tau(y,2t) \le K\tau(y,t)$forall$y \in \Omega$ and all $t > 0$. The smallest such K is called the Δ_2-constant of τ (see (Musielak, 1983)).

Lemma 8. (Bennettand, 1988) Let$\tau \in \Phi$and$1 < s \le q(y) \le q^+ < \infty$. Suppose for all $C > 0$ the condition,$\tau(y, Ct) \le C^{q(y)}\tau(y,t)$ holds, where$y \in \Omega$ and $t > 0$. Then a functionτ satisfies the Δ_2-condition, with constant $K = 2^{q^+}$ (for proof see [3]). The following Lemma characterize bounded, sublinear operators from one Musielak-Orlicz spaces to another.

RESULTS

Theorem 1. Let $v(x)$ and $w(x)$ are weights on R^n. Suppose that X_w be a p-convex weighted BFSs for$1 \le p < \infty$ on R^n. Then the inequality,

$$\|\mathcal{M}f\|_{X_w} \le C\|f\|_{L_{p,v}} \quad (1)$$

Holds for every$f \geq 0$ and for all $a \in (0,1)$ if and only if

$$A(a) = \sup_{t>0}\left(\int_{|y|<t}[v(y)]^{-p'}dy\right)^{\frac{a}{p'}}\left\|\chi_{\{|z|>t\}}(.)\left(\int_{|y|<|.|}[v(y)]^{-p'}dy\right)^{\frac{1-a}{p'}}\right\|_{X_w} < \infty \quad (2)$$

Moreover, if$C > 0$ is the best possible constant in (1), then

$$\sup_{0<a<1}\frac{p'A(a)}{(1-a)\left[\left(\frac{p'}{(1-a)}\right)^p+\frac{1}{a(p-1)}\right]^{1/p}} \leq C \leq M \inf_{0<a<1}\frac{A(a)}{\left(\frac{1}{1-a}\right)^{-\frac{1}{p'}}}.$$

Proof of Theorem 1. Sufficiency. Passing to the polar coordinates, we have

$$m(y) = \left(\int_{|z|<|y|}[v(z)]^{-p'}dz\right)^{\frac{a}{p'}} = \left(\int_0^{|y|}s^{n-1}\left(\int_{|\xi|=1}[v(s\theta)]^{-p'}d\theta\right)ds\right)^{\frac{a}{p'}}$$

where $d\xi$ is the surface element on the unit sphere. Obviously, $m(y) = m(|y|)$, i.e.,$m(y)$ is a radial function. Applying Hölder's inequality for $L_p(R^n)$ spaces and after some standard transformations, we have

$$\|\mathcal{M}f\|_{X_w} = \left\|\sup_{x\in|y|<|.|}\frac{1}{||y|<|.||}w(.)\int_{|y|<|.|}f(y)dy\right\|_X = \left\|\sup_{x\in|y|<|.|}\frac{1}{||y|<|.||}w(.)\int_{|y|<|.|}[f(y)m(y)v(y)][m(y)v(y)]^{-1}dy\right\|_X$$

$$\leq \left\|w(.)\|fmv\|_{L_p(|y|<|.|)}\|[mv]^{-1}\|_{L_{p'}(|y|<|.|)}\right\|_X = \left\|\left\|w(.)fmv\chi_{\{|y|<|.|\}}(y)\|[mv]^{-1}\|_{L_{p'}(|y|<|.|)}\right\|_{L_p}\right\|_X$$

$$= \left\|wfmv\chi_{\{|.|<|x|\}}(.)\|[mv]^{-1}\|_{L_{p'}(|.|<|x|)}\right\|_{X[L_p]}$$

and we have

$$\left\|wfmv\chi_{\{|.|<|x|\}}(.)\|[mv]^{-1}\|_{L_{p'}(|.|<|x|)}\right\|_{X[L_p]} \leq M\left\|wfmv\chi_{\{|.|<|x|\}}(y)\|[mv]^{-1}\|_{L_{p'}(|.|<|x|)}\right\|_{L_{p[X]}}$$
$$= M\left\|\left\|w(.)fmv\chi_{\{|y|<|.|\}}(y)\|[mv]^{-1}\|_{L_{p'}(|y|<|.|)}\right\|_X\right\|_{L_p}$$
$$= M\left\|fmv\left\|w(.)\chi_{\{|y|<|.|\}}(y)\|[mv]^{-1}\|_{L_{p'}(|y|<|.|)}\right\|_X\right\|_{L_p}$$

By switching to polar coordinates and after some calculations, we get

$$\|[mv]^{-1}\|_{L_{p'}(|y|<|x|)} = \left(\int_{|y|<|x|}[m(|y|)v(y)]^{-p'}dy\right)^{\frac{1}{p'}} = \left(\int_0^{|x|}r^{n-1}[m(r)]^{-p'}\left[\int_{|\zeta|=1}[v(r\theta)]^{-p'}d\theta\right]dr\right)^{\frac{1}{p'}}$$
$$= \left(\int_0^{|x|}\left[\int_0^r s^{n-1}\left(\int_{|\zeta|=1}[v(s\theta)]^{-p'}d\theta\right)ds\right]\left(\int_{|\zeta|=1}[v(r\theta)]^{-p'}d\theta\right)^{-a}r^{n-1}\,dr\right)^{\frac{1}{p'}}$$
$$= \frac{1}{(1-a)^{\frac{1}{p'}}}\left(\int_0^{|x|}\frac{d}{dr}\left\{\left(\int_0^r s^{n-1}\left(\int_{|\zeta|=1}[v(s\theta)]^{-p'}d\theta\right)ds\right)^{1-a}\right\}dr\right)^{\frac{1}{p'}}$$
$$= \frac{1}{(1-a)^{\frac{1}{p'}}}\left(\int_0^{|x|}s^{n-1}\left(\int_{|\zeta|=1}[v(s\theta)]^{-p'}d\theta\right)ds\right)^{\frac{1-a}{p'}} = \frac{1}{(1-a)^{\frac{1}{p'}}}\left(\int_{|z|<|x|}[v(z)]^{-p'}dz\right)^{\frac{1-a}{p'}}$$

Therefore from the condition (2), we obtain

$$\left\| fmv \left\| w(.)\chi_{\{|y|<|.|\}}(y) \|[mv]^{-1}\|_{L_{p'}(|y|<|.|)} \right\|_X \right\|_{L_p}$$

$$= \frac{1}{(1-a)^{\frac{1}{p'}}} \left\| fv \left[m \left\| \chi_{\{|.|>|y|\}} \left(\int_{|z|<|.|} [v(z)]^{-p'} dz \right)^{\frac{1-a}{p'}} \right\|_{X_w} \right] \right\|_{L_p} \leq \frac{A(a)}{(1-a)^{\frac{1}{p'}}} \|fv\|_{L_p}.$$

Thus, forall $a \in (0,1)$, $\|\mathcal{M}f\|_{x_w} \leq M \frac{A(a)}{(1-a)^{\frac{1}{p'}}} \|f\|_{L_{p,v}}$

Necessity. Let $f \in L_{p,v}(R^n)$, $f \geq 0$ and the inequality (1) is valid. Wechoosethe test function as

$$f(x) == \frac{p'}{1-a}[g(t)]^{-\frac{a}{p'}-\frac{1}{p}} v^{-p'}(x)\chi_{\{|x|<t\}}(x) + [g(|x|)]^{-\frac{a}{p'}-\frac{1}{p}} v^{-p'}(x)\chi_{\{|x|>t\}}(x)$$

where $t > 0$ is a fixed number and

$$g(t) = \int_{|y|<t} v^{-p'}(y)dy = \int_0^t s^{n-1} \left(\int_{|\eta|=1} v^{-p'}(s\eta)d\eta \right) ds$$

It is obvious that

$$\frac{dg}{dt} = t^{n-1} \int_{|\eta|=1} v^{-p'}(t\eta)d\eta$$

Again by switching to polar coordinates, from the right hand side of inequality (1) we get that

$$\|f\|_{L_{p,v}} = \left[\int_{|x|<t} \left(\frac{p'}{1-a}\right)^p [g(t)]^{-a(p-1)-1} v^{-p'}(x)dx + \int_{|x|>t} [g(x)]^{-a(p-1)-1} v^{-p'}(x)dx \right]^{1/p}$$

$$= \left[\left(\frac{p'}{1-a}\right)^p [g(t)]^{a(1-p)} + \int_t^\infty r^{n-1} [g(r)]^{-a(p-1)-1} \left(\int_{|\zeta|=1} v^{-p'}(s\theta)d\theta \right) dr \right]^{1/p}$$

$$= \left[\left(\frac{p'}{1-a}\right)^p [g(t)]^{a(1-p)} - \frac{1}{a(p-1)} \int_t^\infty \frac{d}{dr} [g(r)]^{-a(p-1)} dr \right]^{1/p}$$

$$= \left[\left(\frac{p'}{1-a}\right)^p [g(t)]^{a(1-p)} - \frac{1}{a(p-1)} \left([g(t)]^{-a(p-1)} - \left[\int_{R^n} v^{-p'}(y)dy \right]^{-a(p-1)} \right) \right]^{1/p}$$

$$\leq \left[\left(\frac{p'}{1-a}\right)^p + \frac{1}{a(p-1)} \right]^{1/p} [g(t)]^{-\frac{a}{p'}}$$

$$= \left[\left(\frac{p'}{1-a}\right)^p + \frac{1}{a(p-1)} \right]^{1/p} [m(t)]^{-1}.$$

After some calculations, from the left hand side of inequality (1), we have

$$\|\mathcal{M}f\|_{X_w} = \left\| \sup_{x\in|y|<|.|} \frac{1}{\big||y|<|.|\big|} \int_{|y|<|.|} f(y)dy \right\|_{X_w} \geq \left\| \chi_{\{|y|>t\}} \sup_{x\in|y|<|.|} \frac{1}{\big||y|<|.|\big|} \int_{|y|<|.|} f(y)dy \right\|_{X_w}$$

$$= \left\| \chi_{\{|y|>t\}} \left(\frac{p'}{1-a} \int_{|y|<|t|} [g(t)]^{-\frac{a}{p'}-\frac{1}{p}} v^{-p'}(y)dy + \int_{t<|y|<|.|} [g(t)]^{-\frac{a}{p'}-\frac{1}{p}} v^{-p'}(y)dy \right) \right\|_{X_w}$$

$$= \left\| \chi_{\{|y|>t\}} \left(\frac{p'}{1-a} [g(t)]^{\frac{1-a}{p'}} + \int_t^{|.|} r^{n-1} [g(r)]^{-\frac{a}{p'}-\frac{1}{p}} \left(\int_{|\eta|=1} v^{-p'}(r\eta)d\eta \right) dr \right) \right\|_{X_w}$$

$$= \left\| \chi_{\{|y|>t\}} \left(\frac{p'}{1-a} [g(t)]^{\frac{1-a}{p'}} + \frac{p'}{1-a} \int_t^{|.|} \frac{d}{dr} [g(r)]^{\frac{1-a}{p'}} dr \right) \right\|_{X_w}$$

$$= \left\| \chi_{\{|y|>t\}} \left[\frac{p'}{1-a} [g(t)]^{\frac{1-a}{p'}} + \frac{p'}{1-a} \left([g(.)]^{\frac{1-a}{p'}} - [g(t)]^{\frac{1-a}{p'}} \right) \right] \right\|_{X_w}$$

$$= \frac{p'}{1-a} \left\| \chi_{\{|y|>t\}} [g(.)]^{\frac{1-a}{p'}} \right\|_{X_w}$$

Hence,

$$\frac{p'}{1-a} \left[\left(\frac{p'}{1-a} \right)^p + \frac{1}{a(p-1)} \right]^{-1/p} [g(t)]^{\frac{a}{p'}} \left\| \chi_{\{|.|>t\}} [g(.)]^{\frac{1-a}{p'}} \right\|_{X_w} \leq \frac{p' A(a)}{(1-a) \left[\left(\frac{p'}{1-a} \right)^p + \frac{1}{a(p-1)} \right]^{1/p}} \leq C$$

For all $a \in (0,1)$. Theorem 1 is proved.

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REFERENCES

Akin L. and Zeren Y. 2017. "Approximation to generalized Taylor derivatives by integral operator families", *MSU J. Sci*. Volume 5, Issue 2, p 421-423.

Bandaliev, R.A. 2008. On an inequality in Lebesguespace with mixed norm and with variable sum ability exponent, *Matem. Zametki*, 84, no. 3, 323-333 (in Russian): *English translation: in Math. Notes,* 84(2008), no. 3, 303-313. MR2473750 (2010k:46030).

Bandaliev, R.A. 2012. Criteria of two-weighted inequalities for multi dimensional Hardy type operator in weighted Musielak-Orliczspaces and some application, *Math. Nachr*.

Bandaliev, R.A. 2012. On a two-weight criteria for multidimensional Hardy type operator in p-convex Banach function spaces and some applicatio, *Ukrainian Mathematical Journal*.

Bennett C. and R. Sharpley, 1988. Interpolation of operators, *PureAppl. Math. 129, Academic Press,* MR 0928802 (89e:46001)

Cochran J.A. and C.S. Lee, 1984. Inequalities related to Hardy's and Heinig's, *Math. Proc. Camb. Phil. Soc*., 96, 1-7. MR 86g: 26026.

Diening L. and S. Samko, 2007. Hardyinequality in variable exponent Lebesgue spaces, *Frac. Calc. Appl. Anal.*10, no. 1, 1-18.

Diening L., P. Harjulehto, P. Hasto and M. Ruzicka, 2011. Lebesgue and Sobolev spaces with variable exponents, *Springer LectureNotes, v.*2017*, Springer-Verlag, Berlin.*

Lindenstrauss J. and L. Tzafriri, 1979. Classical Banach spaces II, *Ergebnisse der Mathematikundihrer Grenzgebiete* 97, *Springer-Verlag, Berlin-Heidelberg-New York,* MR 81c:46001

Lomakina E. and V. Stepanov, 1998. On the Hardy-type integral operator in Banach function spaces, *Publicacions Matema`tiques*, 42, 165-194 .

Luxemburg, W.A.J. 1955. Banachfunctionspaces, *Thesis, Delfi*.

Mamedov, I.F., Zeren, Y., Akın, L. 2017. Compactification of weighted Hardyoperator in variable exponent Lebesgue spaces, *Asian Journal of Mathematics and Computer Research,* 17(1): 38-47.

Musielak, J. 1983. Orliczspacesandmodularspaces, *LectureNotes in Math. (1034), Springer-Verlag, Berlin-Heidelberg-New York*, MR 724434 (85m:46028)

Quinsheng, L. 1993. Twoweight Φ-inequalitiesfortheHardyoperator, Hardy-Littlewoodmaximaloperatorandfractionalintegrals, *Proc. Amer. Math. Soc*., 118, no. 1, 129142. MR1279455(99j: 36037)

Schep, A. 1995. Minkowski's integral inequality for function norms, Operator theory. Operator theory in functions paces and Banachlattices. 75, *Oper. TheoryAdv. Appl*., 299-308.

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