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RESEARCH ARTICLE

PRODUCTS OF CONJUGATE SECONDARY NORMAL MATRICES

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ABSTRACT

Studying metabolites In this paper, the properties of the products of conjugate secondary normal (con-s-normal) matrices are developed, their relation, in a sense, to s-normal matrices is considered and further results concerning s-normal products are obtained.
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INTRODUCTION

Let C_{nxn} be the space of nxn complex matrices of order n. For $A \in C_{n\times n}$, let A^T, $\overline{A}$, A^*, A^S, A^θ and A^{-1} denote the transpose, conjugate, conjugate transpose, secondary transpose, conjugate secondary transpose and inverse of matrix A respectively. The conjugate secondary transpose of A satisfies the following properties such as

$\left(A^\theta\right)^\theta = A, \left(A+B\right)^\theta = A^\theta + B^\theta, \left(AB\right)^\theta = B^\theta A^\theta$. Etc

Definition 1

A matrix $A \in C_{n\times n}$ is said to be normal if $AA^* = A^*A.$

Definition 2

A Matrix $A \in C_{n\times n}$ is said to be conjugate normal (con-normal) if $AA^* = \overline{A^*A}.$

Definition 3

A matrix $A \in C_{n\times n}$ is said to be secondary normal (s-normal) if $AA^\theta = A^\theta A.$

Definition 4

A matrix $A \in C_{n\times n}$ is said to be unitary if $AA^* = A^*A = I.$

Definition 5

A matrix $A \in C_{n\times n}$ is said to be s-unitary if $AA^\theta = A^\theta A = I.$

Definition 6 [2]

A matrix $A \in C_{n\times n}$ is said to be a conjugate secondary normal matrix (con-s-normal) if $AA^\theta = \overline{A^\theta A}$ where $A^\theta = \overline{A}^S$. **. . . (1)**

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Properties of Con-s-Normal Matrices

Theorem 1

A matrix A is con-s-normal iff there exists an s-unitary matrix U such that UAU^S is a direct sum of non-negative real numbers and of $2x2$ matrices of the form: $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ where a and b are non negative real numbers.

Proof

Let A be con-s-normal where $A = P+Q$ where $P = P^S$ and $Q = -Q^S$. Then $A\overline{A}^S = A^S\overline{A}$ gives $(P+Q)\left(\overline{P}^S+\overline{Q}^S\right)=\left(P^S+Q^S\right)\left(\overline{P}+\overline{Q}\right)$ or $(P+Q)\left(\overline{P}+\overline{Q}\right)=(P-Q)\left(\overline{P}+\overline{Q}\right)$ and so:
$P\overline{P}+Q\overline{P}-P\overline{Q}-Q\overline{Q} = P\overline{P}-Q\overline{P}+P\overline{Q}-Q\overline{Q}$ or $Q\overline{P}-P\overline{Q}$.

There exists a s-unitary U such that $USU^S = D$ is a secondary diagonal matrix with real, non–negative elements. Therefore $UQU^S\overline{U}\ \overline{P}\ \overline{U}^S = U\,PU^S\overline{U}\ \overline{Q}\overline{U}^S$ or $WD = D\overline{W}$ where $W = -W^S$. Let U be chosen so that D is such that $d_i \geqq d_j \geqq 0$ for $i<j$ where d_i is the i^{th} secondary diagonal element of D. $W = (t_{ij})$, where $t_{ji} = -t_{ij}$ then $t_{ij}\,d_j = d_i\,\bar{t}_{ij}$, for $j>i$, and 3 possibilities may occur : if $d_j = d_i \neq 0,$ then t_{ij} is real; if $d_j = d_i = 0,$ t_{ij} is arbitrary (through $W=-W^S$ still holds); and if $d_j \neq d_i$, then $t_{ij}=0$ for if $t_{ij} = a+ib$ then $(a+ib)\,d_j = d_i\,(a-ib)$ and $a(d_j - d_i) = 0$ implies $a=0$ and $b\left(d_i + d_j\right)=0$ implies $d_i = -d_j$ (which is not possible since the d_i are real and non–negative and $d_j \neq d_i$) or $b=0$ so $t_{ij}=0$. So if $UPU^S = d_1I_1 \oplus d_2I_2 \oplus \ldots \oplus d_kI_k$ where $\oplus$ denotes direct sum, then $UQU^S = T_1 \oplus T_2 \oplus \ldots . \oplus T_k$ where $Q_i = -Q_i^S$ is real and $Q_K = -Q_K^S$ is complex iff $d_k = 0$. For each real Q_i there exists a real-s-orthogonal matrix V_i so that $V_iT_iV_i^S$ is direct sum of zero matrices and matrices of the form $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ where b is real (Bellman, 1960). If $Q_K = -Q_K^S$ is complex, there exists a complex s-unitary matrix V_k such that $V_kQ_kV_kQ$ is a direct sum of matrices of the form **[3]** so that if $V = V_1 \oplus V_2 \oplus \ldots \oplus V_k$ then $VUPU^SV^S = D$ and $VUQ^SU^S = F$ the direct sum. Therefore $V\,UAU^S\,V^S = D + F$ this is the desired form. If A and B are two con-s-normal matrices such that $A\overline{B} = B\overline{A}$ then A and B can be simultaneously brought into the above secondary normal form under the same U (with a generalization to a finite number) but not conversely; if A is con-s-normal , $A\overline{A}$ is s-normal in the usual sense, but not conversely; and if A is con-s-normal and $A\overline{A}$ is real, there is a real secondary orthogonal matrix which gives the above form. Among properties of con-s-normal matrices not obtained but of subsequent use are the following:

- A is con-s-normal iff $A = HU = UH^S$ where H is s-hermitian and U is s-unitary.
- For if $A = HU$ is a polar form of A, then $\overline{U}^S HU = K$ is such that $A=HU=UK$ and if $A\overline{A}^S = A^S A,$ then $H^2 = \left(K^S\right)^2$ and since this is an s-hermitian matrix with non-negative roots, $H = K^S$ and $A = HU = UH^S$. The converse is immediate. This same result may be seen as follows. If $UAU^S = F$ is the s-normal form in **Theorem 1**, $F = D_rV = VD_r$ where D_r is real secondary diagonal and V is a direct sum of 1's block of the form $\left(a^2+b^2\right)^{-\frac{1}{2}}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ which are s-unitary. Therefore $A = \overline{U}^S D_rU\,\overline{U}^S V\overline{U} = \overline{U}^S V\overline{U}\,U^S D_r\overline{U}$ which exhibits the polar form in another guise.
- A is both s-normal and con-s-normal iff $A = HU = UH = UH^S$ so $H = H^S = \overline{H}^S$ so that H is real.
- If $A = HU = UH^S$ is con-s-normal, then UH is con-s-normal iff $HU^2 = U^2H,$ that is HU^2 is s-normal. For if UH is con-s-normal, $UH = H^SU$ so that $HU^2 = UH^SU = U^2H$; and if $HU^2 = U^2H$, then $HUU = UH^SU = UUH$ or $H^SU = UH$.

- A matrix A is con-s-normal, iff A can be written $A = PW = \overline{W}P$ where $P = P^S$ and W is s-unitary. If A is con-s-normal, form the above $A = \overline{U}^S F\overline{U} = \overline{U}^S D_r \overline{U} U^S V\overline{U} = PW = \overline{U}^S VU\overline{U}^S D_r \overline{U} = \overline{W}P$ where $P = \overline{U}^S D_r \overline{U}$ s-
- symmetric and $W = U^S V\overline{U}$ is s-unitary. Conversely, if $A = PW = \overline{W}P$, $A\overline{A}^S = PW\overline{W}^S\overline{P}^S = A^S\overline{A} = P^S\overline{W}^S\overline{P}$.

  Note that if B is con-s-normal, and if $B=PU$ where $P = P^S$ and U is s-unitary, it does not necessarily follow that $B = \overline{U}P$; but it possible to find on P_1 and U_1 such that $B = P_1U_1 = \overline{U}_1P_1$ holds. This may be seen as follows. If $B=PU$ is con-s-normal, Let V bes-unitary such that $VPV^S = D$ is secondary diagonal, real and non negative, so that $VBV^S = VPV^S\overline{V}UV^S = DW$ is con-s-normal from which $DW\overline{W}^S\overline{D} = W^S D^S \overline{D}\overline{W}$ or since D is real, $WD^2 = D^2W$ and $WD=DW$ since D is non-negative. Then $B = \left(\overline{V}^S DV\right)\left(V^S W\overline{V}\right) = PU = \left(\overline{V}^S WV\right)\left(\overline{V}^S D\overline{V}\right)$ which is not necessarily equal to $\overline{U}P = \left(\overline{V}^S\overline{W}V\right)\left(\overline{V}^S D\overline{V}\right)$ However, if $D = r_1I_1 \oplus r_2I_2 \oplus \ldots \oplus r_kI_k$, $r_i > r_j$ for $i > j$, then $W = W_1 \oplus W_2 \oplus \ldots \oplus W_K$.

Since each W_i is s-unitary, it is con-s-normal and there exist s-unitary X_i so that $X_iW_iX_i^S = F_i$ is in the real s-normal form of **Theorem 1** if $X = X_1 \oplus X_2 \oplus \ldots \oplus X_k$, then $XVBV^SX^S = XDWX^S = DXWX^S = DF = FD$ where $F = F_1 \oplus F_2 \oplus \ldots \oplus F_k$.

So

$$B = \left(\overline{V}^S\,\overline{X}^S D\overline{X}\,\overline{V}\right)\left(V^S X^S F\overline{X}\,\overline{V}\right)$$

$$= \left(\overline{V}^S\,\overline{X}^S FXV\right)\left(\overline{V}^S\,\overline{X}^S D\overline{X}\,\overline{V}\right) = P_1U_1 = \overline{U}_1^S P_1 \; and$$

$$P_1 = \overline{V}^S\,\overline{X}^S D\overline{X}\,\overline{V} \neq \overline{V}^S D\overline{V} = P \; and$$

$$U_1 = V^S X^S F\overline{X}\overline{V} \neq V^S W\overline{V} = U.$$

Products of s-Normal Matrices

If A, B and AB are s-normal matrices then BA is s-normal; a necessary and sufficient condition that the product AB, of two s-normal matrices A and B be s-normal is that each commute with the s-hermitian polar matrix of the other. First a generalization of this theorem is obtained here and then an analog for the con-s-normal case is developed.

Theorem 2

Let A be an s-normal matrix. Then AB and BA are s-normal iff $\left(\overline{A}^S A\right)B = B\left(A\overline{A}^S\right)$ and $\left(\overline{B}^S B\right)A = A\left(B\overline{B}^S\right)$. (In a sense, the latter condition might be described as stating that each matrix is s-normal relative to the other).

Proof

If AB and BA are s-normal, Let U be a unitary matrix such that $UA\overline{U}^S = D$ is secondary diagonal. $d_i\overline{d}_i \geqq d_j\overline{d}_j \geq 0$ for $i < j$, and let $UB\overline{U}^S = B_1 = \left(b_{ij}\right)$. From $AB\overline{B}^S\overline{A}^S = \overline{B}^S\overline{A}^S AB$ it follows that $DB_1\overline{B}_1^S\overline{D} = \overline{B}^S\overline{D}DB_1$; by equating secondary diagonal elements it follows that $\sum_{j=1}^{n} d_i\overline{d}_i b_{ij}\overline{b}_{ij} = \sum_{j=1}^{n} d_j\overline{d}_j b_{ji}\overline{b}_{ji}$ for $i=1,2\ldots n$. Similarly from $BA\overline{A}^S\overline{B}^S = \overline{A}^S\overline{B}^S BA$ follows $B_1D\overline{D}\overline{B}_1^S = \overline{D}\overline{B}_1^S B_1D$ and $\sum_{j=1}^{n} d_j\overline{d}_j b_{ij}\overline{b}_{ij} = \sum_{j=1}^{n} \overline{d}_i d_i \overline{b}_{ji} b_{ji}$. Let $i=1$ in each of these equations So that $\sum_{j=1}^{n} d_1\overline{d}_1 b_{1j}\overline{b}_{1j} = \sum_{j=1}^{n} d_j\overline{d}_j b_{j1}\overline{b}_{j1}$ and $\sum_{j=1}^{n} d_j\overline{d}_j b_{1j}\overline{b}_{1j} = \sum_{j=1}^{n} \overline{d}_1 d_1 \overline{b}_{j1} b_{j1}$ from which follows

$$\sum_{j=1}^{n}\left(d_1\overline{d}_1-d_j\overline{d}_j\right)b_{1j}\overline{b}_{1j}=\sum_{j=1}^{n}\left(d_j\overline{d}_j-d_1\overline{d}_1\right)d_{j1}\overline{b_{j1}}$$

so that $$\sum_{j=1}^{n}\left(d_1\overline{d}_1-d_j\overline{d}_j\right)\left(b_{1j}\overline{b_{1j}}+b_{j1}\overline{b_{j1}}\right)=0.$$

Let $d_1\overline{d_1}=d_2\overline{d_2}=...=d_l\overline{d_l}>d_{l+1}d_{l+1}$, then $b_{1j}\overline{b_{1j}}+b_{j1}\overline{b_{j1}}=0$ for $j=l+1,l+2,...n$ since $d_1\overline{d_1}-d_j\overline{d_j}$ is zero or positive and is latter for $j>l.$ So $b_{1j}=0$ and $b_{j1}=0$ for $j=l+1,l+2,...n.$ For *i=2,.....l* in turn it follows that $b_{ij}=0$ and $b_{ji}=0$. For *i=1,2,....l* and for *j=l+1,l+2....n*. Let $UA\overline{U}^S=D=r_1D_1\oplus r_2D_2\oplus...\oplus r_SD_S$ where the r_i are real $r_i>r_j$ for $i<j$ and the D_i are s-unitary Then by repeating the above process it follows that $UB\overline{U}^S=B_1=C_1\oplus C_2\oplus...\oplus C_S$ is conformable to *D*.

It follows from the given conditions that $r_iD_iC_i\overline{C_i}^S\overline{D_i}r_i=\overline{C_i}^S\left(r_i\overline{D_i}\right)\left(D_ir_i\right)C_i$ and $C_ir_iD_i\overline{D_i}r_i\overline{C_i}^S=r_i\overline{D_i}\,\overline{C_i}^SC_iD_ir_i$ or that $D_iC_i\overline{C_i}^S=\overline{C_i}^SC_iD_i$ and $D_iC_i\overline{C_i}^S=\overline{C_i}^SC_iD_i$ if $r_i>0$. If

$r_s=0$, D_s is arbitrary insofar as *D* is concerned and so may be chosen so that $D_SC_S\overline{C_S}^S=\overline{C_S}^SC_SD_S$ in which case D_s may not be secondary diagonal. But whether or not this is done, it follows that $DB_1\overline{B_1}^S=\overline{B_1}^SB_1D$ and that $B_1D\overline{D}^S=\overline{D}^SDB_1$ so that $A\left(B\overline{B}^S\right)=\left(\overline{B}^SB\right)A$ and $B\left(A\overline{A}^S\right)=\left(\overline{A}^SA\right)B$. The converse is immediate. It may be noted that if the roots of *A* are all distinct in absolute value, *B* must be s-normal. The following further clarifies the situation.

Theorem 3

Let $A=LW=WL$ be the polar form of the s-normal matrix *A*. Then *AB* and *BA* are s-normal iff $B=N\overline{W}^S$ where *N* is s-normal and $LN=NL$.

Proof

In the proof of the above theorem, let $C_i=H_iU_i=U_iK_i$ be polar forms of the C_i. Then $\overline{U_i}^S H_iU_i=K_i$ so that $\overline{U_i}^S C_i\overline{C_i}^SU_i=\overline{C_i}^SC_i$ or $\overline{U_i}^S C_i\overline{C_i}^S=\overline{C_i}^SC_i\overline{U_i}^S$. Also, from the above $D_iC_i\overline{C_i}^S=\overline{C_i}^SC_iD_i$.

Let $R_i=\overline{D_i}\,\overline{U_i}^S$ then $R_iC_i\overline{C_i}^S=\overline{D_i}\,\overline{U_i}^SC_i\overline{C_i}^S=\overline{D_i}\,\overline{C_i}^SC_i\overline{U_i}^S=C_i\overline{C_i}^S\overline{D_i}\,\overline{U_i}^S=C_i\overline{C_i}^SR_i$ where R_i is s-unitary (if $r_s=0$, D_S may be chosen $=\overline{U_S}^S$ as described above). So $R_iH_i^2=H_i^2R_i$ and since H_i has positive or zero roots, $R_iH_i=H_iR_i$ and so $H_i\overline{R_i}^S=\overline{R_i}^SH_i$. Then $A=\overline{U}^SDU=\overline{U}^SD_rU\overline{U}^SD_UU=LW=WL$ and

$$\begin{aligned}B&=\overline{U}^SB_1U=\overline{U}^S\left(C_1\oplus C_2\oplus...\oplus C_S\right)U\\&=\overline{U}^S\left(H_1U_1\oplus H_2U_2\oplus...\oplus H_SC_S\right)U\\&=\overline{U}^S\left(H_1\overline{R_1}^S\overline{D_1}\oplus H_2\overline{R_2}^S\overline{D_2}\oplus...\oplus H_S\overline{R_S}^S\overline{D_S}\right)U\\&=NWC^{-S}\end{aligned}$$

where $N=\overline{U}^S\left(H_1\overline{R_1}^S\oplus H_2\overline{R_2}^S\oplus...\oplus H_S\overline{R_S}^S\right)U$ (which is s-normal since the s-hermitian H_i and s-unitary $\overline{R_i}^S$ commute) and $\overline{W}^S=\overline{U}^S\left(\overline{D_1}\oplus\overline{D_2}\oplus...\oplus\overline{D_S}\right)U.$ It is evident that $LN=NL$.

Conversely, if $A = LW = WL$ and $B = N\overline{W}^S$ as described, then $AB = WLN\overline{W}^S$ which is obviously s-normal as is $BA = N\overline{W}^S WL = NL.$ It is easy seen that $B = N\overline{W}^S$ is s-normal iff $N\overline{W}^S = \overline{W}^S N$. if $B = N\overline{W}^S = (HR)\overline{W}^S$ is con-s-normal; then $B = H\left(R\overline{W}^S\right) = \left(R\overline{W}^S\right)H^S = RH\overline{W}^S$ (form property **(a)**) so $\overline{W}^S H^S = H\overline{W}^S$ or $WH = H^S W$ and $W\left(B\overline{B}^S\right) = \left(\overline{B}^S B\right)W.$

If A is s-normal and B is con-s-normal then AB is s-normal, it does not necessarily follow that BA is s-normal though it can occur. For example, if $B = HU = UH^S$ is

con-s-normal and if $A = \overline{U}^S$ then $AB = \overline{U}^S UH^S$ and $BA = HU\overline{U}^S = H$ are both s-normal. But the following is an example in which AB is s-normal but not BA. Let $B = HU = UH^S$ be

con-s-normal but not s-normal (i.e, H is not real by property **(b)**) and let H be non-singular. Let $A = H^{-1}$ is s-hermitian (So s-normal) and not con-s-normal (since H^{-1} is not real). Then $AB = H^{-1}HU = U$ is s-normal if BA were also s-normal, then by the above theorem $\left(\overline{A}^S A\right)B = B\left(A\overline{A}^S\right)$ and $\left(\overline{B}^S B\right)A = A\left(B\overline{B}^S\right)$. But $\left(\overline{B}^S B\right)A = \left(H^s\right)^2 H^{-1}$ and $A\left(B\overline{B}^S\right) = \left(\overline{H}^{-1}\right)\left(H^2\right)$ and if these were equal, $\left(H^s\right)^2 = H^2$ would follow which means that $H^2 = \left(H^s\right)^2 = \left(\overline{H}^S\right)^2$ so that H^2 real. But this is not possible for if $H = VD\overline{V}^S$ where D is secondary diagonal with positive real elements (since H is non singular), then $H^2 = VD^2\overline{V}^S = \overline{V}DV^S$ if H^2 is real so that $V^S VD^2 = D^2 V^S V$ so $V^S VD = DV^S V$ so $VD\overline{V}^S = \overline{V}DV^S = H$ is real which contradicts the above assumption.

Theorem 4

If A and B are con-s-normal and if AB is s-normal then BA is s-normal.

Proof

Let U be a s-unitary matrix such that $UAU^S = F$ is the s-normal from described in **Theorem 1** and where $F\overline{F}^S = FF^S = r_1^2 I_1 \oplus r_2^2 I_2 \oplus \ldots \oplus r_k^2 I_k$ which is real s-diagonal with $r_1^2 > r_2^2 > \ldots > r_k^2 \geqq 0$ There r_i^2 may be either the squares of secondary diagonal elements of F or they may arise when matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ are squared. Assume that any of the latter whose r_i^2 are equal are arranged first in a given block followed by any secondary diagonal elements whose square is the same r_i^2.

Let $\overline{U}B\overline{U}^S = B_1$ which is con-s-normal and then $UAU^S\overline{U}B\overline{U}^S = F\,B_1$ is s-normal Let V be the s-unitary matrix.

$$V = \begin{bmatrix} \sqrt{1/2} & i\sqrt{1/2} \\ i\sqrt{1/2} & \sqrt{1/2} \end{bmatrix}$$

Then the following matrix relation holds, independent of a and b:

$$V\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\overline{V}^S = \begin{bmatrix} a-bi & 0 \\ 0 & a+bi \end{bmatrix}$$

Let $F = F_1 \oplus F_2 \oplus \ldots \oplus F_k$ where the direct sum is conformable to that of $F\overline{F}^S$ given above $\left(i.e,\ F_i\overline{F_i}^S = r_i^2 I_i\right)$ and consider $F_1 = G_1 \oplus G_2 \oplus \ldots \oplus G_i \oplus r_i I$ where each G_i is 2x2 as described above and I is an identity matrix of proper size. Let $W_1 = V \oplus V \oplus \ldots \oplus V \oplus I$ be conformable to F_1; define W_i for each F_i in like manner and let $W = W_1 \oplus W_2 \oplus \ldots \oplus W_K$. If

$r_k = 0, W_k = I$. Then $WF\overline{W}^S = D$ is complex secondary diagonal, where if d_i is the i^{th} secondary diagonal element $d_i \overline{d}_i \geqq d_{i+1} \overline{d}_{i+1}$. Then $W\left(UAU^S\right)\overline{W}^S W\left(\overline{U}B\overline{U}^S\right)\overline{W}^S = \left(WF\overline{W}^S\right)\left(WB_1\overline{W}^S\right) = DB_2$ is s-normal for $B_2 = WB_1\overline{W}^S$ (or $B_1 = \overline{W}^S B_2 W$). Since B_1 is con-s-normal, $B_1\overline{B}_1^S = B_1^S\overline{B}_1$ so that $\overline{W}^S B_2 W\overline{W}^S \overline{B}_2^S W = W^S B_2^S \overline{W} W^S \overline{B}_2 W$ or that $B_2\overline{B}_2^S WW^S = WW^S B_2^S \overline{B}_2$. Now VV^S is a matrix of the form $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$. So that WW^S is a direct sum of matrices of this form and one's.

Let $B_2 = (b_{ij})$ and consider $\overline{\left(WW^S\right)}^S B_2\overline{B}_2^S\left(WW^S\right) = B_2^S\overline{B}_2$. Let $B_2\overline{B}_2^S = \left(c_{ij}\right)$, $B_2^S\overline{B}_2 = \left(f_{ij}\right)$. c_{ij} and f_{ij} are identifiable with the b_{ij}, both matrices being s-hermitian. Consider two cases:

- If $d_1\overline{d}_1 = d_j\overline{d}_j$ for all j (where d_j is the j^{th} secondary diagonal element of D), then $D=KD_u$ where D_u is s-unitary diagonal. Since $WFB_1\overline{W}^S = DB_2 = KD_uB_2 = D_u\left(KB_2\right)$ is s-normal, then $\overline{D}_u\left(D_uB_2K\right)D_u = B_2D = WB_1F\overline{W}^S$ is s-normal, as is $B_1F = \overline{U}B\overline{U}^S UAU^S$ so BA is s-normal.
- If $d_1\overline{d}_1 \neq d_j\overline{d}_j$ for some j, let $d_1\overline{d}_1 = d_2\overline{d}_2 \ldots = d_l\overline{d}_l$ $for\ 1 \leq l < n$ (so that $d_l\overline{d}_l > d_{l+1}\overline{d}_{l+1}$).
- Suppose $F_1 = G_1 \oplus G_2 \oplus r_1 I_1$ where I_1 is the 2x2 matrix (The general case will be seen to follow from this example). From $\overline{\left(WW^S\right)}^S B_2\overline{B}_2^S\left(ww^S\right) = B_2^S\overline{B}_2$ and the fact that $W_1 = V \oplus V \oplus I_1$ it follows that $C_{11}=f_{22}, C_{22}=f_{11}, C_{33}=f_{44}, C_{44}=f_{33}, C_{55}=f_{55}, C_{66}=f_{66}$ (and $\overline{C}_{12} = f_{12}. \overline{C}_{34} = f_{34}$ etc) there equalities supply the following relation (where the summation is over $i=1$ to n).

$$C_{11} = \sum b_{1i}\overline{b}_{1i} = \sum b_{i2}\overline{b}_{i2} = f_{22};$$
$$C_{22} = \sum b_{2i}\overline{b}_{i2} = \sum b_{i1}\overline{b}_{i1} = f_{11};$$
$$C_{33} = \sum b_{3i}\overline{b}_{3i} = \sum b_{i4}\overline{b}_{i4} = f_{44};$$
$$C_{44} = \sum b_{4i}\overline{b}_{4i} = \sum b_{i3}\overline{b}_{i3} = f_{33};$$
$$C_{55} = \sum b_{5i}\overline{b}_{5i} = \sum b_{i5}\overline{b}_{i5} = f_{55};$$
$$C_{66} = \sum b_{6i}\overline{b}_{6i} = \sum b_{i6}\overline{b}_{i6} = f_{66};$$

DB_2 is s-normal so that the following relations also hold:

$$d_1\overline{d}_1, \sum b_{1i}\overline{b}_{1i} = \sum d_i\overline{d}_i b_{i1}\overline{b}_{i1};$$
$$d_1\overline{d}_2, \sum b_{2i}\overline{b}_{2i} = \sum d_i\overline{d}_i b_{i2}\overline{b}_{i2};$$
$$d_3\overline{d}_3, \sum b_{3i}\overline{b}_{3i} = \sum d_i\overline{d}_i b_{i3}\overline{b}_{i3};$$
$$d_4\overline{d}_4, \sum b_{4i}\overline{b}_{4i} = \sum d_i\overline{d}_i b_{i4}\overline{b}_{i4};$$
$$d_5\overline{d}_5, \sum b_{5i}\overline{b}_{5i} = \sum d_i\overline{d}_i b_{i5}\overline{b}_{i5};$$
$$d_6\overline{d}_6, \sum b_{6i}\overline{b}_{6i} = \sum d_i\overline{d}_i b_{i6}\overline{b}_{i6};$$

Since $d_1\overline{d}_1 = d_2\overline{d}_2$ on combining the first 2 relation in each of these sets,

$d_1\overline{d}_1\left(\sum b_{1i}\overline{b_{1i}} + \sum b_{2i}\overline{b_{2i}}\right) = d_1\overline{d}_1\left(\sum b_{i1}\overline{b}_{i1} + \sum b_{i2}\overline{b}_{i2}\right) = \sum d_i\overline{d_i}\left(b_{i1}\overline{b_{i1}} + b_{i2}\overline{b_{i2}}\right)$ so that

$\sum\left(d_1\overline{d}_1 - d_i\overline{d}_i\right)\left(b_{i1}\overline{b_{i1}}+b_{i2}\overline{b}_{i2}\right)=0$ $d_1\overline{d}_1 = d_j\overline{d}_j$ for $j=1,2...6$ but for j beyond 6, $d_1\overline{d}_1 = d_j\overline{d}_j > 0$ or $b_{i1}\overline{b_{i1}}+b_{i2}\overline{b_{i2}}=0$ or $b_{i1}=0$ *and* $b_{i2}=0$ for $i=7,8....n$ similarly, $b_{i3}=0$ and $b_{i4}=0$ for $i>6$ the third relation in each set give $b_{i5}=0$ and $b_{i6}=0$ for $i>6$.

On adding all 6 relation in the first set,

$$\sum_{i,j=1}^{6} b_{ij}\overline{b}_{ij} + \sum_{i=1}^{6}\sum_{j=7}^{n} b_{ij}\overline{b}_{ij} = \sum_{i,j=1}^{6} b_{ij}\overline{b}_{ij} + \sum_{i=7}^{n}\sum_{j=1}^{6} b_{ij}\overline{b}_{ij}$$

and on canceling the first summations on each side,

$$\sum_{i=1}^{6}\sum_{j=7}^{n} b_{ij}\overline{b}_{ij} = \sum_{i=7}^{n}\sum_{j=1}^{6} b_{ij}\overline{b}_{ij}.$$

But the right side is zero from the above, so the left side is 0 and so $b_{ij}=0$ for $i=1,2...6$ and $j>6$. From this it is evident that this procedure may be repeated and that if $D=r_1D_1 \oplus r_2D_2 \oplus ... \oplus r_kD_k$. Where the D_i are s-unitary and the r_i non–negative real, as above, then $B_2=C_1 \oplus C_2 \oplus ... \oplus C_k$ Conformable to D then $r_iD_iC_i$ is s–normal so $\overline{D}_i^S\left(D_iC_ir_i\right)D_i = C_ir_iD_i$ is s-normal so B_2D is s-normal. So B_1F and so $\overline{U}B\overline{U}^S UAU^S$ and BA.

Theorem 5

If A and B are con-s-normal then AB is s-normal iff $\overline{A}^S AB = BA\overline{A}^S$ and $AB\overline{B}^S = \overline{B}^S BA$ (ie, iff each is s-normal relative to the other).

Proof

If AB is s-normal, from the above $\overline{D}^S DB_2 = B_2D\overline{D}^S$ so that $\overline{F}^S FB_1 = B_1F\overline{F}^S$ or $\overline{A}^S AB = BA\overline{A}^S$.
Similarly DB_2 is s-normal, $DB_2\overline{B}_2^S\overline{D} = \overline{B}_2^S\overline{D}DB_2$ *so* $DB_2\overline{B}_2^S = \overline{B}_2^S B_2D$ or $FB_1\overline{B}_1^S = \overline{B}_1^S B_1F$ or $AB\overline{B}^S = \overline{B}^S BA$. the converse is directly verifiable.

Theorem 6

Let A and B be con-s-normal, if AB is s-normal, then $A=LW=WL^S$ (with L s-hermitian and W s-unitary) and $B = N\overline{W}^S$. Where N is s-normal and $L^S N = NL^S$; and conversely.

Proof

As above, let $UAU^S = F = \overline{W}^S DW = \overline{w}^S D_r w\overline{w}^S D_u w$ where D_r and D_u are the

s-hermitian and s-unitary polar matrices of D) and $\overline{U}B\overline{U}^S = B_1 = \overline{W}^S B_2W = \overline{W}^S\left(C_1 \oplus ... \oplus C_K\right)W$. As in the proof of **Theorem 3** if follows that for all i, $D_iC_i\overline{C}_i^S = \overline{C}_i^S C_iD_i$ and $\overline{U}_i^S C_i\overline{C}_i^S = \overline{C}_i^S C_i\overline{U}_i^S$ with U_i as defined there, so that when $R_i = \overline{D}_i\overline{U}_i^S$ (where D, here, $=r_1D_1 \oplus r_2D_2 \oplus ... \oplus r_kD_k$ as earlier) then $C_i = H_iU_i = H_i\overline{R}_i^S\overline{D}_i$ *with* $H_iR_i = R_iH_i$.

Then since, $WD_r=D_rW$, $UAU^S = \overline{W}^S D_r w\overline{W}^S D_u w = D_r\left(\overline{W}^S D_u w\right)$ and

$$A = \left(\overline{U}^S D_r U\right)\left(\overline{U}^S\overline{w}^S D_u w\overline{U}\right) = LX$$
$$= \left(\overline{U}^S\overline{w}^S D_u w\overline{U}\right)\left(U^S D_r\overline{U}\right) = XL^S$$

with $L = \overline{U}^S D_r U$ s-hermitian and $X = \overline{U}^S\overline{w}^S D_u w\overline{U}$ s-unitary.

Also, $\overline{U}B\overline{U}^S = \overline{w}^S \left(H_1 \overline{R}_1^S \overline{D}_1 \oplus H_2 \overline{R}_2^S \overline{D}_2 \oplus ... \oplus H_k \overline{R}_k^S \overline{D}_k \right) w = N_1 Y$

Where $N_1 = \overline{w}^S \left(H_1 \overline{R}_1^S \oplus H_2 \overline{R}_2^S \oplus ... \oplus H_k \overline{R}_k^S \right) w$ is s-normal and $Y = \overline{w}^S \left(\overline{D}_1 \oplus \overline{D}_2 \oplus ... \oplus \overline{D}_k \right) w$ is s-unitary; then $B = U^S N_1 Y U = \left(U^S N_1 \overline{U} \right) \left(U^S Y U \right) = N \overline{X}^S$.

Where $N = U^S N_1 \overline{U}$ is s-normal and $\overline{X}^S = U^S Y U = U^S \overline{W}^S \overline{D_u} w U$. Also

$L^S N = N L^S$ since $D_r N_1 = N_1 D_r, \overline{D}_r N_1 = N_1 \overline{D}_r$ so $\left(\overline{U} \overline{L} U^S \right) \left(\overline{U} N U^S \right) = \left(\overline{U} N U^S \right) \left(\overline{U} \overline{L} U^S \right)$ so $L^S N = N L^S$.

The converse is immediate.

Products of Con-s-Normal Matrices

It is possible if A is s-normal and B con-s-normal that AB is con-s-normal. For example, any con-s-normal matrix $C=HU=UH^S$ is such a product with $A=H$ and $B=U$. Or if $C=HU=UH^S$ and $A=H$, then $AC=H^2U=HUH^S=U(H^S)^2$ is con-s-normal. The following theorems clarify this matter.

Theorem 7

If A is s-normal and B is con-s-normal then AB is con-s-normal iff

$AB\overline{B}^S = B\overline{B}^S A$ and $\overline{B}A\overline{A}^S = A^S \overline{A}\overline{B} (or \overline{B} A A^S = \overline{A}^S A B)$.

(If one were to define N is s-normal with respect to M" to mean $N\overline{N}^S M = M\overline{N}^S N$ and Q is con-s-normal with respect to P to mean $PQ\overline{Q}^S = Q^S \overline{Q} P$ the above theorem would say that if A is s-normal and B is con-s-normal then AB is con-s-normal iff (con-s-normal) B is s-normal with respect to A and (s-normal) A is con-s-normal with respect to $\overline{B}$).

Proof

If the latter condition hold, then; $(AB)\left(\overline{AB}\right)^S = AB\overline{B}^S \overline{A}^S = B\overline{B}^S A\overline{A}^S$ and $(AB)^S \left(\overline{AB}\right) = B^S A^S \overline{A}\ \overline{B} = B^S \overline{B} A \overline{A}^S$ which are equal.

Conversely, let AB be con-s-normal and let $UA\overline{U}^S = D = d_1 I_1 \oplus d_2 I_2 \oplus ... \oplus d_k I_k$ where $d_i \overline{d}_i > d_j \overline{d}_j,\ i > j.$

$$Let\ \ U B^S U^S = B_1 = (bij),$$

$$if\ (AB)\left(\overline{AB}\right)^S = AB\overline{B}^S \overline{A}^S = AB^S \overline{B}\ \overline{A}^S = (AB)^S \left(\overline{AB}\right)$$

$$= B^S A^S \overline{A}\,\overline{B} = B^S \overline{A} A^S \overline{B}\ ,$$

then $\left(UA\overline{U}^S \right) \left(UB^S U^S \overline{U}\,\overline{B}\,\overline{U}^S \right) \left(U\overline{A}^S \overline{U}^S \right) = \left(UB^S U^S \right) \left(\overline{U}\,\overline{A} U^S \overline{U} A^S U^S \right) \left(\overline{U}\,\overline{B}\,\overline{U}^S \right)$

So that $DB_1 \overline{B}_1^S \overline{D}^S = B_1 \overline{D} D \overline{B}_1^S$.

Equating secondary diagonal elements on each side of this relation, we get $\sum_{j=1}^{n} d_i \overline{d}_i b_{ij} \overline{b}_{ij} = \sum_{j=1}^{n} d_j \overline{d}_j b_{ij} \overline{b}_{ij}$, $i=1,2,...n$

or

$$\sum_{j=1}^{n} \left(d_i \overline{d}_i - d_j \overline{d}_j \right) b_{ij} \overline{b}_{ij} = 0\ .$$

Let $d_1 \overline{d}_1 = d_2 \overline{d}_2 = ... d_l \overline{d}_l > d_{l+1} \overline{d}_{l+1}$ then $b_{ij} = 0$ for $i=1,2...l$ and $j = l+1, l+2...n$ since B_l is con-s-normal,

$\sum_{j=1}^{n} b_{ij}\overline{b}_{ij} = \sum_{j=1}^{n} b_{ji}\overline{b}_{ji}$ for $i = 1,2,...n$ on adding the first l of these equation and canceling, $b_{ij} = 0$ for $i = l+1, l+2...n$ and $j = 1,2,...,l$. In this manner if $D = r_1 D_1 \oplus r_2 D_2 \oplus ... \oplus r_t D_t$ with $r_i > r_{i+1}$ and D_i s-unitary, then $B_1 = C_1 \oplus C_2 \oplus ... \oplus C_t$ conformable to D.

Since $r_i D_i \overline{D_i}^S r_i \overline{C_i}^S = r_i^2 C_i^S = C_i^S r_i^2 = C_i^S r_i D_i \overline{D_i}^S r_i$, for all i, $D\overline{D}^S B_1^S = B_1^S D\overline{D}^S$ and so $\overline{U}^S D\overline{D}^S U\overline{U}^S B_1^S \overline{U} = \overline{U}^S B_1^S \overline{U} U^S D\overline{D}^S \overline{U}$ or $A\overline{A}^S B = BA^S\overline{A}$ or $\overline{A}^S AB = BA^S\overline{A}$ or $A^S\overline{A}\overline{B} = \overline{B}A\overline{A}^S$.

Also, $D\left(B_1\overline{B_1}^S\overline{D}^S\right) = B_1\overline{D}D\overline{B_1}^S = \overline{D}D\overline{B_1}^S = D\left(\overline{D}B_1\overline{B_1}^S\right)$ so that $C_i\overline{C_i}^S\left(r_i\overline{D_i}\right) = \left(r_i\overline{D_i}\right)C_i\overline{C_i}^S$ for $i = 1,2...t$. (if $r_t = 0$, this is still true and D_t may be chosen to be identity matrix). Therefore $B_1\overline{B_1}^S\,\overline{D}^S = \overline{D}^S B_1\overline{B_1}^S$ and $UB^S U^S\,\overline{U}\,\overline{B}\,\overline{U}^S U\overline{A}^S\overline{U}^S = U\overline{A}^S\overline{U}^S UB^S U^S\overline{U}B_1\overline{U}^S$ so $B^S\overline{B}\overline{A}^S = \overline{A}^S B^S\overline{B}$ *or* $AB^S\overline{B} = B^S\overline{B}A$.

Corollary 1

Let A be s-normal, B con-s-normal; if AB is con-s-normal, then $B\overline{A}$ is con-s-normal, and conversely.

Proof

From the above, $UA\overline{U}^S UBU^S = DB_1^S$ is con-s-normal, and if $D = D_r D_u$, D_r real and D_u s-unitary, then since $\overline{D_u} = \overline{D_u}^S$, $\overline{D_u}\left(DB_1^S\right)\overline{D_u} = D_r B_1^S\overline{D_u} = B_1^S D_r\overline{D_u} = B_1^S\overline{D}$ is con-s-normal, as are $U\,BU^S\,\overline{U}\overline{A}\,U^S$ and $B\overline{A}$. Reversing the steps proves the converse.

If A is s-normal and B is con-s-normal, $B\overline{A}$ is con-s-normal iff AB is con-s-normal, iff $\left(B^S\,\overline{B}\right)A = A\left(B\overline{B}^S\right)$ and $\left(A^S\,\overline{A}\right)\overline{B} = \overline{B}\left(A\overline{A}^S\right)$. Therefore if A is s-normal B is con-s-normal BA is con-s-normal iff $\left(B^S\overline{B}\right)\overline{A} = \overline{A}\left(B\overline{B}^S\right)$ and $\left(\overline{A}^S A\right)\overline{B} = \overline{B}\left(\overline{A}A^S\right)$ that is replace A by $\overline{A}$ in the proceeding or $\left(\overline{B}^S B\right)A = A\left(\overline{B}B^S\right) = A\left(\overline{B}^S B\right)$ and $\left(\overline{A}^S A\right)\overline{B} = \overline{B}\left(\overline{A}A^S\right)$, thus exhibiting the fact that when AB is con-s-normal, BA is not necessarily so.

Theorem 8

If $A = LW = WL$ is s-normal and $B = KV = VK^S$ is con-s-normal (where L and K are s-hermitian and W and V are s-unitary) then AB is con-s-normal iff $LK = KL$, $LV = VL^S$ and $WK = KW$.

Proof

If the three relations in the theorem hold, then $AB = LWKV = LKWV$, and $AB = WLKV = WKLV = WKVL^S = WVK^S L^S = WV\left(LK\right)^S$ is con-s-normal since LK is s-hermitian and WV is s-unitary.

Conversely, Let $A = \overline{U}^S DU = \left(\overline{U}^S D_r U\right)\left(\overline{U}^S D_u U\right) = LW$ and

$$B = \left(\overline{U}^S B_1^S\overline{U}\right) = \left(\overline{U}^S K_1 U\right)\left(\overline{U}^S V_1\overline{U}\right) = KV = VK^S$$

where K_1 and V_1 are s-hermitian and s-unitary and direct sums conformable to B_1^S and *D*. A direct check shows that $LK = KL$ and $LV = VL^S$, also $WK = \overline{U}^S D_u K_1 U = \overline{U}^S K_1 D_u U = KW$ since $D_u B_1 \overline{B_1}^S = B_1 \overline{B_1}^S D_u$ implies $D_u K_1 = K_1 D_u$. A sufficient condition for the simultaneous reduction of *A* and *B* is given by the following:

Theorem 9

If *A* is s-normal, *B* is con-s-normal and $AB = BA^S$, then $WA\overline{W}^S = D$ and $WB^S W = F$, the s-normal form of **Theorem 1**, where W is an s-unitary matrix; also *AB* is con-s-normal.

Proof

Let $UA\overline{U}^S = D$ secondary diagonal and $UBU^S = B_2$ which is con-s-normal. Then $AB = BA^S$ implies $DB_2 = UA\overline{U}^S UBU^S = UBU^S \overline{U} A^S U^S = B_2 D^S = B_2 D.$

Let $D = C_1 I_1 \oplus C_2 I_2 \oplus .... \oplus C_K I_K$. Where the C_i are complex and $C_i \neq C_j$ for $i \neq j$ and $B_2 = C_1 \oplus C_2 \oplus .... \oplus C_K$ let V_i be s-unitary such that $V_i C_i V_i^S = F_i$ the real s-normal form of **Theorem 1**, and let $V = V_1 \oplus V_2 \oplus ... \oplus V_k$.

Then $VUA\overline{U}^S \overline{V}^S = D,\ VUBU^S V^S = F = a$ direct sum of the F_i.

Also, $AB = BA^S$ implies $B^S A^S = AB^S$ and so $AB\overline{B}^S \overline{A}^S = AB^S \overline{B}\overline{A}^S = B^S A^S \overline{A}\overline{B} = (AB)^S \left(\overline{AB}\right).$

It is also possible for the product of two s-normal matrices *A* and *B* to be con-s-normal if $Q = HU = UH^S$ is con-s-normal and if $A = U$ and $B = H$ this is so or if $KV = VK^S$ is con-s-normal and if $A=UK=KU$ is s-normal with *K* s-hermitian and V and *U* s-unitary, for $B = V$, $AB = (UK)V = K(UV) = (UV)K^S$ con-s-normal. But if in the first example, U^2H is not s-normal then HU is not con-s-normal so that *BA* is not necessarily con-s-normal though *AB* is. When *A* alone is s-normal an analog of Theorem 2 can be obtained which states the following: if *A* is s-normal, then *AB* and AB^S are con-s-normal iff $AB\overline{B}^S = B^S \overline{B} A$, $B\overline{B}^S A = AB^S \overline{B}$ and $\overline{B} A\overline{A}^S = A^S \overline{A}\overline{B}$. (The proof is not included here because of its similarity to that above) when *B* is con-s-normal, two of these conditions merge into one in Theorem 7. It is possible for the product of two con-s-normal matrices to be con-s-normal but no such simple analogous necessary and sufficient conditions as exhibited above are available. This may be seen as follows two non-real complex commutative matrices $P = P^S$ and $Q = Q^S$ can form a con-s-normal (and non-real s-symmetric) matrix *PQ* which need not be

s-normal. Then two s-symmetric matrices $X = \begin{bmatrix} -i & -i \\ i & -i \end{bmatrix}$ $Y = \begin{bmatrix} 2i & 0 \\ 0 & 2i \end{bmatrix}$ are such that *XY=Z* is real, s-normal and con-s-normal

(s-symmetric). Finally if *U* and V are two complex s-unitary matrices of the same order, they can be chosen so *UV* is non-real that is complex, s-normal and con-s-normal. If $A = P \oplus X \oplus U$ and $B = Q \oplus Y \oplus V$ $AB = PQ \oplus XY \oplus UV$ where *A* and *B* are con-s-normal as in *AB.*

(s-symmetric). A simple inspection of these matrices shows that relations on the order of $\left(B^S \overline{B}\right) A = A\left(B\overline{B}^S\right) = \left(B\overline{B}^S\right) A$ and $\left(A^S \overline{A}\right)\overline{B} = \left(A\overline{A}^S\right)\overline{B} = \overline{B}\left(A\overline{A}^S\right)$ do not necessarily hold; these are sufficient, however, to guarantee that *AB* is con-s-normal (as direct verification from the definition).

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